Nelson's Work on Logic and Foundations and Other Reflections on Foundations of Mathematics

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Abstract

This paper starts by discussing Nelson's philosophy of mathematics, which is a blend of mathematical formalism and a radical constructivism. As such, it makes strong assertions about the foundations of mathematic and the reality of mathematical objects. We then offer our own suggestions for the definition of mathematics and the nature of mathematical reality. We suggest a second characterization of mathematical reasoning in terms of common sense reasoning and argue its relevance for mathematics education.

Nelson's philosophy is the foundation of his definition of predicative arithmetic. There are close connections between predicative arithmetic and the common theories of bounded arithmetic. We prove that polynomial space (PSPACE) predicates and exponential time (EXPTIME) predicates are predicative.

We discuss Nelson's formalist philosophies and his unpublished work in automatic theorem checking.

This paper was begun with the plan of discussing Nelson's work in logic and foundations and his philosophy on mathematics. In particular, it is based on our talk at the Nelson meeting in Vancouver in June 2004. The main topics of this talk were Nelson's predicative arithmetic and his unpublished work on automatic theorem proving. However, it proved impossible to stay within this plan. In writing the paper, we were prompted to think carefully about the nature of mathematics and more fully formulate our own philosophy of mathematics. We present this below, along with some discussion about mathematics education.

Much of the paper focuses on Nelson's philosophy of mathematics, on how his philosophy motivates his development of predicative arithmetic, and on his unpublished work on computer assisted theorem proving. We also discuss the

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nature of mathematical reality and Nelson's views and our own views on the nature of mathematics. Predicative arithmetic and, more generally, Nelson's philosophy of mathematics, are closely related to Nelson's development of internal set theory and nonstandard analysis, but this connection is not pursued in the present paper; for this, the reader may consult G. Lawler's paper in this volume.

The main body of the paper is written to be accessible to a mathematician with little knowledge of logic. The paper begins with a quick overview of three of the main philosophies of mathematics: formalism, platonism, constructivism. We then present the basic ideas behind Nelson's predicative arithmetic, a framework that he has put forward as being the correct general setting for mathematical reasoning. The next section, Section 4, discusses our own views of mathematics. We give one definition of mathematics, and then a second characterization of mathematics and discuss some implications for mathematics education. After that, we return to Nelson's work on automatic theorem proving, and then the main body of the paper concludes with some quotes from Nelson's writings.

Two appendices include extra material. The first appendix gives a new, weaker base theory Q^- for predicative arithmetic. The second appendix proves that exponential time computability is predicative.

1 Platonism, constructivism and formalism

In our talk [1], we described Nelson's philosophy of mathematics as being "radical constructivism." Afterward, Nelson suggested that he thinks of himself as a "formalist" rather than a "radical constructivist." In fact, both of these labels apply very well to Nelson's philosophy; he is not merely a formalist, he also gives both predicative arithmetic and non-standard analysis very constructive foundations. The reader is warned, however, that we will be expressing our personal opinions of Nelson's philosophy, and Nelson might not always agree with what we say. In particular, in comparison with the present paper, Nelson would probably stress formalism more and constructivism less.

The "platonist" philosophy of mathematics takes the view that the usual objects of mathematic study, such as the integers, the real numbers, functions on the real numbers, etc. — even abstract sets — have some kind of independent existence. The nature of this existence is typically left vague, and the platonist philosophy usually posits that these mathematical objects exist only in some abstract or non-physical sense. Nonetheless, the hallmark of the platonist philosophy is that human mathematicians have direct intuition about or direct perception of mathematical objects. The platonist philosophy holds that our mathematical theorems and constructions are "about" something real.

Most mathematicians are platonists, but there are competing schools of thought. These include the "intuitionists," the "constructivists," and the "formalists."[1] Constructivism was advocated by D. Hilbert as a means to establish

[1]We will not discuss intuitionism at all in this paper.

the consistency of formalizations of theories about the real numbers and set theory. The constructivist philosophy takes the point of view that finite objects, notably the integers, exist in some platonic sense and that finitary combinatorial operations on the integers have a well-defined semantics. A constructivist would generally reject the existence of infinite sets, or at least the existence of a "completed infinity", but would accept the meaningfulness of the concept of an arbitrary integer. The usual convention is that primitive recursive operations are the constructive operations.

Formalism is the viewpoint that mathematics is merely a 'game' that acts on symbols according to a fixed set of rules. For a formalist, the activity of a mathematician consists of manipulating formulas according to fixed rules to generate proofs; for instance, manipulating statements of first-order logic to generate theorems. The formalist rejects the platonist assumption of the existence of mathematical objects; therefore, the semantics, or meaning, of formulas and theorems is not considered to be relevant or, for that matter, even to be defined.

The platonist and formalist philosophies are diametrically opposed. To a platonist, the formalist philosophy would be felt to be a sterile environment of unmotivated reasoning using meaningless symbols. To a formalist, the platonist philosophy would appear to be a misguided — perhaps dangerously misguided — study of objects that do not exist. Needless to say, most mathematicians are closer to the platonist philosophy than the formalist philosophy. If nothing else, the platonist approach has proved to be immensely powerful and fruitful. Nonetheless, there has been trouble in paradise. Hilbert proposed his so-called Hilbert's program to establish at least the formal *consistency* of platonistic reasoning using constructive methods. However, Gödel's incompleteness theorems showed that Hilbert's program was impossible and its goals could not be achieved. In fact, Gödel's theorem showed that constructive methods could not even establish the consistency of the *constructive* theories of arithmetic. Thus Hilbert's program for justifying the use of platonic methods failed completely even for justifying the platonic use of constructive objects.

Nelson takes this failure of Hilbert's program at face value. He not only doubts the platonic existence of the set of all integers, but even doubts the consistency of Peano arithmetic. As we shall see, Nelson does believe, at least implicitly, in the existence of some kind of mathematical infinity and in the meaningfulness of some simple kinds of constructive operations such as integer addition and multiplication. However, he does not accept the set of integers as a given entity or even the reality of all primitive recursive operations. Hence, I use the terminology "radical constructivism" to describe this part of his philosophy.

The radical constructivist philosophy underlies Nelson's predicative arithmetic; predicative arithmetic is a weak formal theory of the integers that is mathematically very similar to the theories of bounded arithmetic of [24, 31, 2]. The original definition of bounded arithmetic, $I\Delta_0$, by Parikh [24] was motivated in part by constructivism and in part by feasible computability (see the survey [3]); however, most researchers in bounded arithmetic adopt only the mathematical trappings of constructivity and very few subscribe to radical constructivism. Sazonov [26, 27, 28], however, advocates a form of radical constructivism.

Along with doubting the existence of the integers, and thereby doubting the existence of a fixed semantics for reasoning about infinite objects, Nelson also maintains a formalist philosophy. To reiterate, this means he maintains that there are no platonic mathematical objects which mathematics is about, rather that mathematics consists purely of formal manipulation of first-order formulas. The paper [17] contains Nelson's most emphatic declaration of the formalist philosophy. A more recent discussion is in [20].

The pure formalist philosophy is usually coupled with a rejection of any non-formal intuition or reasoning. When I was a graduate student, Nelson told me on more than one occasion that his approach to mathematics is purely formal and that he does not have any intuition for what is true or false. At the time I took this to mean he asserted that he did not have intuition for mathematics. I found this surprising since it contradicts our usual experiences in mathematics; indeed, even Nelson's own writing and teaching are full of intuition and motivation in a way that appear to contradict his being a pure formalist. I was, of course, misinterpreting Nelson's statements. Nelson maintains a purely formalist position and denies the existence of platonic mathematical objects; correspondingly, he feels there is no possibility of having intuition about what is true or false, on the grounds that there are no platonic objects to have intuition about. But he strongly asserts that one can have intuition about what theorems are provable and about what mathematical constructions are possible. In a recent emailed personal communication, he says

> ... I admit — proclaim! — the possibility and necessity of intuition about what kinds of formulas can be proved.

To make an analogy, a formalist might view mathematicians as being similar to architects. An architect who is preparing to design and erect a building cannot answer questions like "How many floors are there in your building?" since the design is not complete. Nonetheless the design and construction of the building must respect the physical properties of the building materials, just as a formalist's proof must respect the formal definitions of the object under study and the characteristics of the proof system. Similarly to the architect with a yet-unbuilt building, and for much the same reasons, the formalist will maintain that the mathematical concepts do not satisfy definite properties before the proper mathematical constructions and proofs have been carried out. The constraint of physical building materials in no way precludes an architect's use of intuition or creativity. Analogously, the constraints of formal logic in no way preclude a formalist's having full use of, and need of, intuition and creativity.

For now, we leave formalism aside and turn to trying to give some justification for radical constructivism. In section 3, we shall see how this has motivated Nelson's theories of predicative arithmetic.

2 Radical constructivism

> Now I live in a world in which there are no numbers save those that human beings on occasion construct. Nelson [18, p.7]

One frustrating aspect of trying to understand Nelson's philosophy is that he never makes a clear statement of his reasons for doubting the integers. The clearest quote of his on this issue is as follows:[2]

> The reason for mistrusting the induction principle is that it involves an impredicative concept of number. It is not correct to argue that induction only involves the numbers from 0 to n; the property of n being established may be a formula with bound variables that are thought of as ranging over all numbers. That is, the induction principle assumes that the natural number system is given. A number is conceived to be an object satisfying every inductive formula; for a particular inductive formula, therefore, the bound variables are conceived to range over objects satisfying every inductive formula, including the one in question. [22, p. 1]

Nelson actually raises two objections to the use of induction. Both objections are well-known, although it is not common to apply them to integer induction. The first objection is the observation that induction to an integer n is justified by the fact that it is possible to count to n, or in other words, it is possible to reach n from 0 by applying the successor operation (the "+1" operation) a finite number of times. The objection is raised to the fact that there is a circularity in the definition of "finite". Integers represent finite values, whereas finite values are values that correspond to integers.[3]

Nelson's second and main objection is more subtle; namely, he questions the platonic assumption that there is a set of integers and from this rejects the idea that induction can hold for formulas that quantify over all integers. What he objects to especially is that the integers are, on the first hand, being defined as the set of numbers for which induction is valid, and on the second hand, the formulas for which induction must hold involve quantification over the set of all integers.

Since our description of the second objection probably made no sense on first reading, we try to say it again: One common idea of defining the (finite) integers is to say that the integers are the numbers n for which induction up to n is valid. Nelson objects that this definition of the integers is flawed since the formulas for which induction must hold quantify over the very same of set of integers which is in the process of being defined.

Section 3 will discuss how Nelson formulates a definition of the integers in a way that avoids this second objection. First, however, we shall try to further

[2] See [21] for another account of Nelson's doubt about the integers.

[3] The set theoretic method of defining finite in terms of not being equinumerous with a proper subset is too technical and is not convincing as a foundational definition of the integers.

convince the reader that there could actually be good reasons to doubt the existence of the integers.

2.1 Engendering doubt in the integers

How could a mathematician possibly doubt the integers? Even if one does not accept the strong platonist viewpoint that the integers exist as "actual" entities, surely one should believe in them as a set of mental constructs that have definite properties?

The alternatives are unpalatable. If the integers do not exist, then how could they have fixed properties? For example, consider the question:[4]

> Do there exist odd perfect integers?

Most mathematicians certainly believe that the existence or non-existence of an odd perfect integer should be a definite property of the integers. That is, that odd perfect integers exist, or do not exist, independently of the successes or failures of human efforts in doing mathematics.

In contrast to the widespread surety about properties of integers, reasonable people might agree to doubt the relevance or meaningfulness of set theoretic principles such as the axiom of choice (AC) or the continuum hypotheses CH and GCH. It is plausible that there could be multiple, equally compelling, concepts of "set" and thus no reason to believe that CH or GCH have any independent meaning as a platonic truth (or platonic falsity) about actually existing objects.

If the reader will bear with us, we would like to try to engender some doubt about the platonic existence of the integers.

Suppose, hypothetically perhaps, that you believe beyond doubt in the integers and the existence of the set of all integers in some sense. In this case, the concept of a *set of integers* also surely makes sense. So consider the class of all sets of integers, that is, the powerset, $P(\mathbb{N})$, of the integers. Now, consider the continuum hypothesis question: Is there a subset of $P(\mathbb{N})$ of cardinality strictly between the cardinalities of $\mathbb{N}$ and $P(\mathbb{N})$? This is just a question about relations on $P(\mathbb{N}) \times P(\mathbb{N})$. — That is, it is a question about sets of pairs (binary relations) which are subsets of $P(\mathbb{N}) \times P(\mathbb{N})$.

But then you must believe (beyond doubt — ?) in the meaningfulness of the continuum hypothesis problem. However, this was something one might think reasonable to doubt. In short, we have taken faith in the existence of the integers to justify surety in the meaningfulness of the continuum hypothesis. Since one might reasonably doubt the meaningfulness of CH, one should thus reasonably doubt the existence of the integers.

The above argument hinged on the formation of an infinite powerset, so at this point, we have three options of what to believe and what to doubt. We can hold one of the following positions:

[4] Recall that a perfect integer is one that is equal to the sum of its proper divisors, for instance, $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$. The question stated is open.

1. A platonic belief in the existence of integers and reals, including belief in the meaningfulness of the continuum hypothesis.

2. A platonic belief in the existence of the set of integers, i.e., in the completed infinity; but doubt about the meaningfulness of forming powersets of infinite sets.

3. Doubt about the existence of the set of all integers.

Option 1 is the traditional viewpoint of mathematicians; however, the history of set theory has not been very encouraging to this viewpoint. Option 2 is also a widely held view. However, on closer consideration, option 2 has some problems. In particular, it seems that once one accepts the existence of the integers, one is compelled to accept the existence of at least some sets of integers. And once one accepts the existence of some infinite sets of integers, how can one refuse to accept the existence of the class of all infinite sets of integers? Thus, option 3 is viable as an alternative to option 2.

To be honest, there is a fourth option, but it is perhaps the coward's way out:

4. An agnostic viewpoint which refuses to worry about the issue.

We realize that the arguments above will not appeal to everyone, and that the reader may not buy into radical constructivism and may even have absolutely no doubts about the integers. For these readers, we hasten to point out that Nelson's philosophy has lead him to develop predicative formulations of arithmetic and non-standard analysis which are interesting mathematics in their own right, quite apart from any philosophical motivations. These mathematical developments will be discussed in the next section, which describes the foundations of theories of *predicative arithmetic*. These are weak formal systems of consistency strength <u>much</u> weaker than Peano arithmetic.

3 Nelson's Predicative Arithmetic

Nelson's system of predicative arithmetic[5] is a weak form of arithmetic which avoids platonic assumptions about the existence of an infinite set of integers which satisfy induction. Nelson's predicative arithmetic begins with a vague notion of "integer" which is then refined into a more refined notion of integer.[6] In effect, we start with the assumption that there is an infinite set of "proto-integers" (however, Nelson does not use the terminology "proto-integer"), and

[5]The terminology *predicative arithmetic* as used by Nelson is quite different from the (semi-)constructive concept of predicativity studied by S. Feferman and others. Nelson's predicative arithmetic is at a much lower level of logical complexity.

[6]We follow the usual convention in logic of using "integers" to mean the non-negative integers.

then we create refined notions of "integer" by taking subsets of the proto-integers. The proto-integers are closed under successor, addition and multiplication; furthermore, the proto-integers exist as an infinite set and it is permissible to quantify over the set of all proto-integers. However, the proto-integers are *not* presumed to satisfy induction. For that matter, not even all the usual properties of addition and multiplication, e.g., the commutative law, are presumed to hold.

Nelson's predicative arithmetic theories are defined by defining subsets of the proto-integers that satisfy more and more properties of the integers, starting with basic properties like commutativity and working up to stronger properties including induction for bounded formulas. The intuition is that the process of defining more-and-more refined subsets of proto-integers is evolving towards more-and-more refined concepts of integers. However, in keeping with the radical constructivist philosophy, one does not expect to reach the actual integers in the limit (since, after all, the "actual integers" are not believed to exist) or even to reach a limit (since reaching a limit is an infinite process which is certainly at least as complicated as the integers we are trying to capture).

We quickly sketch the mathematical definition of predicative arithmetic; for more details, consult [22]. To begin, the proto-integers are axiomatized in first-order logic. First-order logic includes the logical connectives

> Boolean connectives: $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies).
>
> Quantifiers: $\forall$ and $\exists$.
>
> Equality: $=$.
>
> Numeric constants and functions: 0, S, $+$ and $\cdot$. S is the successor function $S(x) = x + 1$.

The quantifiers are used with variables x that range over all proto-integers.

The axioms for the proto-integers are just the axioms of R. Robinson's theory Q:

$$\begin{array}{ll}
Sx \neq 0 & x + 0 = x \\
Sx = Sy \rightarrow x = y & x + Sy = S(x + y) \\
x \neq 0 \rightarrow (\exists y)(Sy = x) & x \cdot 0 = 0 \\
 & x \cdot Sy = x \cdot y + x.
\end{array}$$

The theory Q is very weak; for instance, it does not imply the principles $\forall x \forall y(x + y = y + x)$ or $\forall x(0 + x = x)$. Nonetheless, Q is powerful enough to serve as the base theory for Peano arithmetic (PA), which is often defined as Q plus induction for all formulas.

Nelson's Predicative Arithmetic is defined starting with Q and augmenting the theory by (a) extending the language and (b) restricting to refined subsets of the proto-integers where more properties hold. In this way, certain kinds of induction are part of predicative arithmetic. We will briefly illustrate the main ideas behind the formation of Predicative Arithmetic from Q up through showing that $0 + x = x$ is predicative. In order to simplify this, we make the assumption that Q also contains the axioms asserting the associativity of addition and multiplication.[7]

Extensions by definition Initially, only the theory Q is accepted as predicative. The theory Q is strengthened in a series of steps by the introduction of new numeric predicates and functions and the establishment of new axioms. This is an iterative process; at each stage there is a "current" version of predicative arithmetic and subsequent stages are stronger and stronger predicative theories.

The first, and easiest, method of extending predicative arithmetic is via the introduction of predicates and function symbols which are defined in terms of previously introduced symbols. For example, the inequality predicate, $\leq$, is a binary relation and can be defined by

$$x \leq y \quad \Leftrightarrow \quad (\exists z)(x + z = y). \tag{1}$$

Adding the symbol $\leq$ to the language of predicative arithmetic along with its defining axiom (1) yields a *conservative* extension of Q. That is to say, anything that can be expressed (or, proved) in the enlarged language can be expressed (resp., proved) in the original language.

Functions can be conservatively introduced by a similar process. For example, the predecessor function, $P(x) = \max\{0, x - 1\}$, can be defined by

$$P(x) = y \quad \Leftrightarrow \quad S(y) = x \vee (x = 0 \wedge y = 0). \tag{2}$$

In order for this definition to be proper, an existence condition and a uniqueness condition must hold. Letting $M(x, y)$ be the right hand side of (2), the existence condition is $\forall x \exists y M(x, y)$ and the uniqueness condition is $\forall x \forall y \forall z (M(x, y) \wedge M(x, z) \rightarrow y = z)$. These conditions must be provable in the already introduced version of predicative arithmetic.

The predicativity of $\mathbf{0} + x = x$. The property $\forall x(0 + x = x)$ does not follow from the theory Q; nonetheless, it can be taken as predicative by defining a suitably refined notion of proto-integers. The first idea might be to just consider the subset of proto-integers that satisfy $0 + x = x$, however, this subset might not satisfy the axioms of Q. Instead a more subtle construction is needed. This construction is due independently to Solovay [29] and Nelson[22]; related methods were used much earlier by Gentzen for the ordinals.

Definition A formula $\phi(x)$ is *inductive* provided it has been proved that

$$\phi(0) \ \wedge \ \forall x(\phi(x) \rightarrow \phi(Sx)).$$

[7] There is no need to assume associativity, rather the associativity axioms are established predicatively in [22]. Furthermore, the assumptions that addition and multiplication are total functions can also be dropped, see Appendix A.

By "it has been proved," we mean in the current theory of predicative arithmetic.

Theorem 1 *(Solovay [29],Nelson [22]) Suppose $\phi(x)$ is inductive. Let*

1. *$\phi^1(x)$ be the formula $\forall y(y \leq x \rightarrow \phi(y))$,*
2. *$\phi^2(x)$ be the formula $\forall y(\phi^1(y) \rightarrow \phi^1(y+x))$, and*
3. *$\phi^3(x)$ be the formula $\forall y(\phi^2(y) \rightarrow \phi^2(y \cdot x))$.*

Then, $\phi^3(x)$ defines an initial segment of the integers which is closed under S, $+$ and $\cdot$.

This theorem is intended to be interpreted as a (meta)theorem about predicative arithmetic. More formally, if predicative arithmetic proves that $\phi(x)$ is inductive, then predicative arithmetic proves that

$$\phi^3(y) \wedge x < y \rightarrow \phi^3(x)$$

and

$$\phi^3(x) \wedge \phi^3(y) \rightarrow \phi^3(Sx) \wedge \phi^3(x+y) \wedge \phi^3(x \cdot y).$$

Theorem 2 *Let $\psi(x)$ be the formula $0 + x = x$. Then, $\psi(x)$ is inductive (relative to Q).*

Proof We argue informally in Q. First, $0 + 0 = 0$ holds by an axiom of Q. Second, suppose $0 + x = x$. Then $0 + Sx = S(0 + x) = Sx$. $\square$

Now, consider the set of proto-integers that satisfy ψ^3, that is, the set $\{x : \psi^3(x)\}$, where ψ^3 is defined from ψ as in Theorem 1. Then, by Theorem 1, this set is inductive and closed under addition and multiplication, and it satisfies the axioms in $Q_2 := Q \cup \{\forall x(0 + x = x)\}$.

In this way, we have predicatively justified the axiom $0+x = x$. This permits the axiom to be adjoined to the list of predicative principles, and the enlarged theory Q_2 to be used as the next iteration of predicative arithmetic.

The predicative justification of Q_2 can be viewed either semantically or syntactically. The semantic view is that we started with a set of proto-integers satisfying Q and obtained a definable subset that satisfies Q_2. In this view, the "real" integers are approximated better and better by successively strong theories of predicative arithmetic. The syntactic view is based on consistency and relative interpretation. The above construction shows that Q_2 is *interpretable* in Q, that is that any Q_2-proof can be transformed into a Q-proof by relativizing to the objects that satisfy ψ^3. In particular, if Q is consistent, then so is Q_2.

Definition [22] A theory $T \supset Q$ is *predicative* if it is interpretable in Q.

The main methods for establishing interpretability are the use of extension by definitions and the use of inductive formulas. Indeed, these are the only methods used by Nelson. After establishing $0+x=x$, a host of other principles can be established predicatively, starting with commutativity of addition and multiplication and working up to much more complicated principles. Examples of predicative principles include:

1. Induction on bounded formulas. Bounded formulas may only use quantifiers which are bounded, $\forall x \leq t$ and $\exists x \leq t$. This includes all of bounded arithmetic, $I\Delta_0$ and S_2.

2. Least number principles for bounded formulas.

3. Sequence coding, Gödel numbers for syntactic objects including formulas and proofs. The *smash function*, $\#$,

$$x\#y = 2^{|x|\cdot|y|}$$

where $|x| \approx \log_2(x)$.

4. Metamathematic concepts including consistency and interpretability and the proof of the Gödel incompleteness theorem.

Establishing the predicativity of the above items is the main content of Nelson's book [22]. On the other hand, some principles which are *not* predicative include:

5. The totality of exponentiation: $exp := \forall x \exists y(2^x = y)$.

6. Having an inductive initial segment on which superexponentiation $2 \Uparrow x$ is total. Here, $2 \Uparrow 0 = 1$ and $2 \Uparrow (x+1) = 2^{2 \Uparrow x}$.

7. The Gentzen cut elimination theorem.

8. The consistency of the theory Q.

However, in regard to **5.**, principles that follow from a *finite* number of uses of exponentiation are predicative. This includes the tautological consistency of Q and the bounded consistency of Q. More generally, Wilkie and Paris [31] showed that any bounded formula which is a consequence of $Q + exp$ is predicative; however, their proof of this fact is impredicative since it depends on the cut elimination theorem.

Connections with computational complexity. As we just discussed, predicative arithmetic includes induction for all bounded formulas and the totality of the smash function. It thus includes the theory of bounded arithmetic $I\Delta_0 + \Omega_1$ and the equivalent theory S_2. The present author's own work in bounded arithmetic [2] was based in part on Nelson's work on predicative arithmetic and also in part on earlier work of Parikh, Dimitracopolous, Paris, Wilkie, Wilmers, and Pudlák. Much of the motivation for bounded arithmetic comes from its close connection to computation complexity; the theories S_2^1 and T_2^i have particularly close connections to low-level complexity classes [2].

Via its inclusion of bounded arithmetic, predicative arithmetic also has close connections to computational complexity. However, predicative arithmetic is properly stronger than bounded arithmetic. Indeed, every exponential-time function is predicative. What this means is that it is possible to form a predicative theory with function symbols for exponential time functions and which has axioms fully characterizing the functions. These axioms are universal closures of bounded formulas. This construction is sketched in more detail in Appendix B. Since the polynomial space functions are a subset of the exponential time functions, the polynomial space computable functions are also predicative.

On the other hand, the superexponential function is not predicative.

It is also possible to give a predicative development of parts of real analysis, at least up through standard results on integration. Ko and Friedman [16, 15] showed that polynomial space computability is sufficient for the definition of integration (more precisely, they showed that the counting class $\#P$ is sufficient). However, they only considered computability, not provability, so this did not say anything per se about predicatively. J. Hook in his 1983 Ph.D. thesis [11] under Nelson developed a predicative version of real analysis with the additional assumption that exponentiation is not total. More recently, Fernandes and Ferreira [6] have given a predicative treatment of parts of real analysis within the framework of reverse mathematics for bounded arithmetic. They show explicitly that their theory of real analysis is interpretable in Q.

3.1 Nelson's predicative philosophy

Nelson apparently proposed his predicative arithmetic as a way of establishing a model for the mathematical universe that encompasses both his formalism and his rejection of the integers. We next discuss to what extent he has succeeded and present two objections to the use of predicative arithmetic as the measure of mathematical reality.

As discussed above, predicative arithmetic starts with the assumption of an initial collection of "proto-integers." These are refined by defining inductive cuts, which are initial segments of the proto-integers that satisfy successively stronger axioms. The motivation behind the proto-integers is that they represent the integers that can be represented in unary notation, perhaps. However, if one is concerned that the integers may not exist as a completed platonic infinity, why should one accept the existence of an infinite set of proto-integers? Since the formation of formulas such as ϕ^2 and ϕ^3 in Theorem 1 requires quantifying over all proto-integers, they implicitly assume the existence of the proto-integers as a completed infinity. Although the existence of a set of proto-integers is a weaker assumption than assuming the existence of the set of platonic integers, it is hardly a more convincing assumption. Nelson himself [private communication] does not not wish to make semantic assumptions about the existence of an infinite set of proto-integers and regards even the consistency of Q, and thereby the consistency of predicative theories, as an open problem.

Nelson gave a compelling and elegant development of much of basic number theory in predicative arithmetic and, as mentioned earlier, this has been extended to a predicative treatment of real analysis. Nelson also argues in [22] that this is the way mathematics *should* be developed, and he raised the following "compatibility problem" as an open question. The compatibility problem asks whether any two predicative principles are compatible. Namely, if ϕ and ψ are both predicative, is their conjunction $\phi \wedge \psi$ also predicative? Unfortunately, Solovay [personal communication] was able to show that the answer is 'No.'.

Solovay's example can be expressed as follows. Define a number x to *log-even* (resp., *log-odd*) if $2 \Uparrow n < x \leq 2 \Uparrow (n+1)$ for some even (resp., odd) number n. The *eventually log-even* property states that there is an x such that all $y > x$ are log-even. Let ϕ be the property

$$\phi \;=\; (\mathit{exp}) \vee (\text{eventually log-even})$$

and ψ be

$$\psi \;=\; (\mathit{exp}) \vee (\text{eventually log-odd}).$$

Then it can be shown that each of ϕ and ψ is predicative; in fact, they are each interpretable in Q with an inductive cut. However, their conjunction is equivalent to the totality of exponentiation and is not interpretable in Q. (Another disproof of the compatibility problem was recently outlined by H. Friedman [8].)

4 What is mathematics? — Two definitions

In this section, we will set aside Nelson's philosophy and present some of our own ideas on the nature of mathematical reality. This will turn out to be a subtle and vague combination of formalism and platonism.

Before saying what mathematics *is*, let us say what it is *not*. In recent years there has been discussion about 'post-modern' ideas about the nature of mathematical reality, e.g., that it is a social activity and mathematics does not have any independent existence (cf. Hersch [10]). An extreme form of post-modernism might assert that mathematical truth depends on the culture or bias of the mathematician. This post-modern idea is completely silly if it is taken as saying that particular mathematical statements could be true for some people and false for others.

On the other hand, the post-modern idea is more-or-less a triviality if it is making the less extreme statement that different individuals or different cultures may make different choices of what kinds of mathematics to study and what kind of evidence is accepted as mathematical proof. There are many examples of the fact that mathematics could be done in different ways. We give three examples here. First, there is a strong tendency for our society to think of the real numbers as being the "real" numbers — we think of the integers and rational numbers as being only a proper subset of the actual numbers and the complex numbers as being an augmentation of the real numbers with "imaginary" numbers. But in actuality, the complex numbers are arguably more natural than the real numbers, and certainly the theory of analytic complex functions of a single variable is a beautiful theory that surpasses the elegance of the theory of real functions. Nonetheless, as a society, we do not properly introduce the complex number until late in college and only to highly technical students. As a second example, consider the fact that we use vectors and dot products and cross products, whereas historically we had the viable alternative of using quaternions instead. For a third example, consider that the real numbers could be formulated using non-standard analysis instead of using the traditional measure theory. Only time will tell whether the non-standard analysis approach will be found to have enough advantages to overcome the societal inertia of using measure theory.

Nonetheless, we wish to reject a definition of mathematics based on human mathematical activity. Rather, we seek a definition of mathematics that transcends the merely social aspect and exposes a more eternal non-social aspect, much as we expect mathematical truths to be eternal.

We also would not be satisfied with a definition of mathematics that merely lists the subjects of mathematical inquiry such as "number", "shape," "pattern," "proof", etc. This kind of definition has several problems: First, it leaves extremely vague what these concepts mean. Second, these same objects can also be the subject of non-mathematical investigations; for instance, an artist might explore properties of geometric shapes and patterns, or a linguist might investigate the etymology of names for numbers. Third, simple concrete ideas like "number," "shape," and "pattern" do not exhaust the objects of mathematical study. Furthermore, as discussed above, the actual objects of mathematics study can change over time.

4.1 Our first definition of mathematics

Without further ado, here is our proposal for a definition of mathematics:

> *Mathematics is the study of objects and constructions, or of aspects of objects and constructions, which are capable of being fully and completely defined.* [4]

Our original statement of this definition was on the foundations of mathematics (FOM) mailing list, and the reader might refer to that for some related discussion. At the time it was met by a modest amount of discussion, plus some opposition.[8] However, this definition is by no means completely new; for example, Polya [25, p.26] states in passing, "Numbers and figures are not the only objects of mathematics. Mathematics is basically inseparable from logic, and it deals with all objects which may be objects of an exact theory." [9]

[8]The definition from [4] was immediately followed by the following sentence: "*A defining characteristic of mathematics is that once mathematical objects are sufficiently well-specified then mathematical reasoning can be carried out with a robust and objective standard of rigor.*" We still believe this statement, but are emphasizing it less in the present discussion.

[9]We are grateful to Khait [14] for bring this quote of Polya's to our attention. For more on Polya's views of the nature of mathematics and mathematical reasoning, see the introduction to [25].

To clarify the definition of mathematics, we consider the case of integers. The integers can be defined in a second order logic in what is generally felt (by platonists) to be a categorical definition, namely, the integers are linearly ordered, each integer has a unique successor, each integer except 0 has a unique predecessor, and every non-empty subset of the integers has a unique least element. Platonists certainly feel this is enough to settle every property about the integers in a definite way. Even for the mathematicians who are not platonists, there is generally agreement that common properties like the existence of odd prefect numbers is a definite fixed property of the integers; in other words, that our conception of integers is already sufficiently well formed so that there can be only one answer to whether odd perfect integers exist.

The situation of sets is more precarious. Reasonable people might feel that the concept of set is not sufficiently well formed for sets to be considered to be fully and completely defined. This is probably true, but certainly many *aspects* of sets are sufficiently well defined, in particular, the parts of set theory used by logicians as the foundations for mathematics. Some of the more problematic questions in set theory such as the continuum hypothesis may depend on aspects of sets that have not yet been fully and completely defined; the continuum hypothesis may need some refinement of the concept of set before it has a chance to settled as being true or false. (We know our formal axioms for set theory are not enough to settle the continuum hypothesis; however, it is unknown whether our intuitive concept of set is sufficiently precise to settle the continuum hypothesis.)

Our definition of mathematics requires only that mathematical objects are "capable of being" fully and completely defined. This phrase was included deliberately. The intent is that (aspects of) objects that have not yet been fully and completely defined are still part of mathematics, provided they are ultimately capable of being fully and completely defined.[10]

Let's consider what our definition means for the formalist and platonist philosophical conceptions. We start with formalism. On the surface, the definition is neutral on the issue of formalism; it says nothing about what kind of methods are used to reason about the "fully and completely defined" objects and constructions. Of course, this means the definition is somewhat opposed to formalism. A formalist would maintain that formal symbol manipulation is the *only* acceptable form of reasoning. Although rigorous deduction and formal reasoning are incredibly powerful for mathematics, we still wish to allow intuitive and common-sense thinking about mathematical objects to be considered mathematics.

Formalism is not completely wrong-headed of course. It is an empirical fact that mathematical reasoning can be carried out with a robust and objec-

[10](This footnote is intended mostly for logicians or other readers who appreciate paradoxes and self-reference.) It is interesting to note that the very definition of mathematics implies that we are not fully and completely defining mathematics. Namely, when we are defining mathematics, we are doing philosophy, not mathematics. If we were able to fully and completely define mathematics, it would make the definition of mathematics itself a mathematical definition!

tive standard of rigor. Disagreements over what constitutes a valid (social) mathematical proof are rare, and when disagreements arise, they can be readily resolved to everyone's satisfaction by fleshing out the proof in more detail. Furthermore, this standard of rigor is, in principle, codifiable in purely formal terms, most notably in first-order logic.[11] In this way, formal reasoning does encompass all of mathematical reasoning. However, we do not agree that formal reasoning is the *only* correct form of mathematical reasoning nor that formal symbol manipulation is the *entire* content of mathematics.

Next, we consider what the definition means for platonism. It is evident that this definition allows for a kind of platonism, but it is a rather unusual kind of platonism. These "fully and completely defined" objects and constructions or aspects of objects and constructions enjoy some kind of mental existence, at least in the trivial sense that we are thinking of them. However, the "aspects of" part is troubling, namely, how could you have an object exist with some fully defined aspects but have some of its other aspects not be fixed? For example, if sets exist, then surely they either do or do not satisfy the continuum hypothesis? How could sets exist if their properties are not fixed?

This issue is not a problem for just mathematics though. Consider the concept of a unicorn. We all have some conception of a unicorn and would admit that unicorns exist in some mental way. However, many aspects of unicorns are vague. For instance: What is a unicorn's gestation period? Is it mortal? Is it susceptible to lice infections? All these are clearly silly questions to ask, especially because a unicorn is supposed to be an idealized animal, not a "real" animal. I maintain however, that mathematical objects should be thought of as existing, like unicorns, in a partial fashion. Furthermore, even though they exist in this sense, mathematical objects may have some aspects fully and completely defined and other aspects not fully defined.

To finish considering this definition for mathematics, it is worth considering how it applies to subjects other than mathematics. Inspired by some comments of S. Simpson on my FOM posting, we consider the case of biology. Suppose a biologist is using mathematical models to study the size of a population over time. Clearly the biologist is using some mathematics, quite possibly even developing new mathematics. But her work is a blend of mathematics and non-mathematics. Many concepts from biology, including core concepts like "population," "individual," and "alive" are not fully and completely defined. They are useful and robust concepts, to be sure, but in exceptional cases, their definition becomes unclear. The mathematical model abstracts from these biological concepts into mathematical concepts; naturally this involves some simplifications and does not fully reflect the biological situation. Thus we see, not surprisingly, that the work with the mathematical model can certainly be considered mathematics, but the application of the mathematical model to biological systems is not mathematics in the sense of our definition.

[11] This is by Gödel's completeness theorem. In most practical cases, the reduction to first-order logic is too detailed and lengthy to carry out by hand, but the theoretical possibility is usually clear.

We also should consider the implications of our definition for the field of mathematics. For practicing researchers in pure mathematics, the definition is completely appropriate and fits well with how mathematics is done in practice. An applied mathematician might be a little less sure of how well it applies to his work. If the emphasis of the applied work is on applications to non-mathematical subjects, then it might not fit the above definition of mathematics. On the other hand, experimental mathematics, for instance studying the efficacy of numerical procedures, would fit into our definition of mathematics.

We add that the definition of mathematics does not involve a value judgement: mathematical reasoning is not meant to be construed as superior to reasoning in other intellectual fields. The specificity of mathematical objects makes possible the high level of rigor in mathematics, and it also allows us to strive for absolute, incontrovertible knowledge about mathematical objects (subject always to human fallibility). However, the specificity of mathematics is also a weakness, since it means that mathematical constructions are merely idealizations of, or approximations to, physical reality. That is to say, by itself, mathematics cannot speak directly of the real, scientific world.

In the next subsection, we say something about the definition of mathematics for people outside the mathematical research community.

4.2 A second, operational definition of mathematics.

The definition of mathematics given above suffers from being too abstract; it doesn't give much guidance as to how one should do mathematics in practice, how one should learn mathematics, nor any way to evaluate the importance of what mathematics is done. To partially make up for this, we give a second descriptive, or operational, definition of mathematics:

> *Mathematical reasoning is a refined form of common sense.*

We call this a "descriptive" or "operational" definition since it is not meant to provide a definition of what is mathematics. Indeed, almost any intellectual activity could be considered to be using refined common sense. Instead, this second definition is meant to describe the nature of mathematical reasoning.

This second definition is not intended to contradict the earlier discussion about the level of rigor in mathematics, rather it says that mathematical rigor is an extension of ordinary common-sense thinking. There are two claims contained in this second definition: (a) Rigorous mathematical thinking is a refined form of common sense; it is not alien to ordinary thinking. (b) Mathematics should "make sense;" i.e., it is not purely formal or rule-based, rather mathematics is about something common-sensical.

Our primary motivation for presenting this second definition is for its implications for mathematics education. Our thesis is that the kind of reasoning used for mathematics is in essence the same as common sense reasoning, albeit tailored for the more abstract and formal environment of mathematics. Common sense reasoning is used in ordinary activities like cooking a meal, washing the dishes, playing with blocks, working on a jigsaw puzzle, and planning a vacation. Mathematical reasoning differs from common sense reasoning in degree, but not in kind. An important consequence for education is that a student should expect mathematics to "make sense"; that is, the student should not be satisfied to learn a set of rote procedures for solving problems, but rather should expect to find that the problems concern definite objects with definite properties and that the procedures should correspond to common sense ideas about about manipulating those objects.

I was prompted to present this second definition in large part by a paper by A. Khait [14] that I received while writing a first draft of this paper. Khait also gives a definition of mathematics based on the quotation from Polya above; Khait did his work independently, but upon later finding my FOM posting felt that his definition was the same as mine [personal communication]. This however is not the case. Khait's definition reads as follows: "Mathematics is a linguistic activity, which is characterized by the association of words with precise meanings." Khait defines "precise" by saying "[For finite mathematics,] computers can serve as the precision criterion: a precise formulation is one that can be translated for a computer. Concerning infinite structures and theories there is no such referee except the public opinion of colleagues." Note that Khait's definition differs from ours in several important ways. First, he defines mathematics in *social* terms as an *activity*, much like Hersch's recent arguments that mathematics should be defined as a social activity. Second, Khait's definition, with its emphases on linguistic activity and on computerization as the measure of precision, leans mostly towards the formalist position, albeit with the difference that Khait does not say symbols are manipulated according to definite rules, rather that words are manipulated that have definite meanings. In contrast, our own definition defined mathematics in terms of its subject matter, and allows for many kinds of reasoning, including linguistic, formal, geometric, and intuitive.

Khait [14] applies his definition to make suggestions for mathematics education, saying that the goal of mathematics education should be to inculcate "an ability to work with words to which precise meanings are assigned." He draws on research of Stanovich-West [30] and others that concludes that individuals have two different styles of thinking. The first style, called "System I" thinking, is intuitive, associative, heuristic, automatic, fast, and compatible with low cognitive capacity. The second style, called "System II," is rational, analytic, controlled, conscious, slow, and demanding of higher cognitive capacity. It is recognized that everyone is capable of System I reasoning, but the further claim is that "experts plus some laypersons" are fluent at System II reasoning, but the majority of the population consists of "untutored individuals" who do not use System II reasoning as effectively. (This is from Khait [14], drawing from Stanovich-West [30]. We are paraphrasing their positions, not endorsing them! The ability to do System II reasoning is largely measured in terms of performance on the SAT tests widely used for college admission in the US.) Khait suggests that the minority group of people who are naturally adept at System II should be educated differently from the majority group who use primarily System I. For the latter group, he advocates improving their System II skills by training them in principles of logic using discrete mathematics examples. Of these students, he says that System II thinking does not come naturally to them, but that many of them nonetheless must learn formal linguistic thinking for their future occupations, which are likely to be computer-related.

This emphasis on the dichotomy of thought processes into System I and System II is questionable. Of course there are different thinking styles, and of course System II thinking benefits greatly from education, but we reject completely the proposition that the majority of people are incapable of thinking effectively in a System II fashion. In fact, System II and System I processes arise in all human activities. Consider riding a bicycle:[12] When first learning to ride a bike with gears and hand brakes, one has to consciously think about keeping one's balance, not going too slow or too fast, changing gears, applying the brakes, etc. This conscious coordination of actions is a kind of System II thinking. Once these skills are learned, they become completely automatic (System I), and it is no longer necessary to think consciously about them at all. In spite of the fact that mathematics is mental instead of physical, one has analogous experiences in mathematics. When first learning a mathematical definition, one has to to think through its implications carefully and working even simple problems about the new definition can require large mental effort. However, once the mathematical concepts have been mastered, the same problems become very simple and intuitively obvious. Once a mathematical concept is fully mastered, it may even be difficult to apply System II reasoning to the concept. (See Khait [12, 13] for similar discussions on how mathematical thinking integrates both systems of thinking.)

In addition, the emphasis on a dichotomy between System I and System II is potentially very harmful to the practice of mathematics education. Both Khait and Stanovich-West talk of educating people to use System II reasoning and the need to train students to use both System I and II thinking; Khait [12] particularly emphasizes that the goal in mathematics education is for students to integrate System I and II thinking. Nonetheless, there is the danger that an overemphasis on the dichotomy between System I and System II thinking could lead to an elitist philosophy that some people, even a majority, cannot handle the more abstract and formal thinking that characterizes System II. It could be extremely detrimental to mathematics education for teachers and students to have these attitudes, since expectations play a large role in performance.

The best application of the theory of System I and System II thinking would not focus solely on improving System II thinking but would instead seek to lead students to combine System I and System II thinking and use both methods to learn and utilize mathematics. Teachers and students should take to heart the maxim that mathematics is a form of common sense. They should accept that mathematics is not arbitrary formal manipulation of symbols but rather is

[12]After writing the first draft of this paper, I discovered that the bicycle analogy is hardly original. Gowers [9, p.32] writes "After one has learned to think abstractly, it can be exhilarating, a bit like suddenly being able to ride a bicycle without having to worry about keeping one's balance.

about precisely defined (aspects of) objects and constructions. Teachers should expect students to not only master rote skills but also to use common sense reasoning about mathematical objects, and in more advanced classes, to phrase their common sense reasoning in formal terms. Most importantly, students should expect and demand that the mathematics they learn make sense and not be merely rote manipulation.

We add that we do not advocate a content-blind approach to mathematics education with the goal of teaching reasoning skills independently of any subject matter, nor do we advocate jettisoning all rote skill training. The best way to teach mathematical reasoning is to introduce meaningful and useful mathematical content. A prime example is the use of geometry as a vehicle for teaching formal mathematical proof in high school, since one can use formal proofs effectively in establishing useful and non-obvious concepts from geometry.

5 Nelson's automated proof checker *qed*

We now return to discussing Nelson's work. Sections 2 and 3 studied mostly the radical constructivist aspects of Nelson's work, but Nelson tells us that he thinks of himself as a formalist rather than a constructivist.

The formalist philosophy is in many ways a very sophisticated philosophy. We do not think the historical development of mathematics could have begun with formalism. Rather, mathematics historically passed through various stages starting with basic concrete facts about quantities and shapes, progressing to the realization that these concrete facts can be abstracted into platonic concepts, and then on to the recognition, via Frege, Russell and Whitehead, Hilbert's program, and Gödel's completeness theorem, that pure logic and formal reasoning are sufficient tools for all mathematical reasoning.

Certainly in recent times, the trend in mathematics has been towards increasing formalization. The modern versions of formalization started in the nineteenth century and reached full maturity in the first half of the twentieth century. The advent of computerization has given further impetus to formalization, both because mathematical logic and formalization have contributed much to the theory of computers and programming languages and because of the possibility that computers can mechanize formal reasoning. A further impetus to formalization is that the heavy use of computerization exposes society to more abstract symbol processing and is arguably predisposing us to think in a computer-like fashion. Whether this trend will continue is unclear. There is even a possibility that, in the next few decades, computers will pick up so much of the burden of calculation and formalization that humans will actually become less adept at formal reasoning rather than more.

Nelson has explored in depth the idea of expressing formalized reasoning in a form that would be amenable to both human understanding and computer verification. In an untitled, unpublished, and unfinished manuscript dated 1993, Nelson revisited the development of predicative arithmetic with an automated proof checker [23]. For this, he wrote an automated proof checker, *qed*, which works directly from text in his TeX files.[13] The system allows theorems to be stated and proved in a formal system with all details automatically checked by the computer. A striking feature is that the same source is used both as input to the *qed* system and to generate a (highly technical) TeX typeset proof. The technical content of the theorems is similar to [22], but now theorems are stated and proved with sufficient formality to be computer-checked.

The proof system *qed* is a deduction proof system (similar to a deduction proof system of Fitch [7], but using very different notations). To illustrate the system *qed*, consider using the axiom

$$\forall x \forall y(x + Sy = S(x + y)) \tag{3}$$

to prove the equality

$$x = 0 + x \to Sx = 0 + Sx. \tag{4}$$

As a precursor to the actual *qed* proof, (4) can be proved from (3) as follows:

$\forall x \forall y(x + Sy = S(x + y))$	(3)	Hypothesis (Axiom)
$x = 0 + x \to Sx = 0 + Sx$	(4)	Goal to be proved
$\{$		Assume its negation
$\quad e = 0 + e \wedge Se \neq 0 + Se$		New variable e for x in $\neg(4)$.
$\quad 0 + Se = S(0 + e)$		Instance of (3).
$\}$		Simple contradiction reached.

For space reasons, we don't give Nelson's definition of a "simple contradiction," but it is a polynomial time test which can detect tautological unsatisfiability in many cases (but cannot detect all cases of unsatisfiability). The proof above shows a single assumption of the negation of the goal, but *qed* also permits more general nesting of assumptions of (negations of) goals.

Nelson adopts a very compact representation of *qed* proofs. The proof given above would be written in *qed* as follows: (x is now used in place of e.)

$$4\{{:}x\ \ 3;0;x\}.$$

Reading the compact proof from left to right, the "4{" means assume the negation of formula (4), the ":x" means replace x by the variable x, the "3" refers to formula (3), the ";0;x" means to substitute 0 and x for the universally quantified variables x and y of (3) and the "}" means the assumption is closed. (The replacement of x with x in the negation of (4) is redundant, but is required by the syntax of *qed*. The variable x is implicitly universally quantified in (4) but is treated as a free variable after the replacement.)

A more complicated example of a proof from Nelson [23] is the following:

Th 158: $x \neq 0 \ \to \ x/x = 1.$

158{:x 113;x;x;1;0 16;x 47;x 130;x 3;$x \cdot 1$ 134;x }

[13] Nelson's software system *qed* should not be confused with the completely independent QED project in automated theorem proving.

For another example, here is another theorem and proof also from [23]. We show verbatim, the statement of the theorem, a human readable form of the proof, and then the corresponding *qed* proof.

Th 159: $x_1 \le x_2 \;\rightarrow\; x_1/y \le x_2/y$.

We have (.1) $y \neq 0$. There is a non-zero u such that $x_2/y + u = x_1/y$, so $x_1 = y \cdot (x_2/y + u) + r_1 = (y \cdot (x_2/y) + (y \cdot u)) + r_1 = y \cdot (x_2/y) + (y \cdot u + r_1)$. There is a z such that $x_2 = x_1 + z$, so that $x_2 = (y \cdot (x_2/y) + (y \cdot u + r_1)) + z = y \cdot (x_2/y) + ((y \cdot u + r_1) + z)$. Consequently, $r_2 = (y \cdot u + r_1) + z = y \cdot u + (r_1 + z)$, so $y \cdot u \le r_2$ and hence $y \cdot u < y$, which is impossible.

159{:x_1:x_2:y .1{ 156;x_1 156;x_2 16;0 } 113;x_1;y;x_1/y;r_1 113;x_2;y;x_2/y;r_2 98;x_2/y;x_1/y 44;x_2/y;x_1/y;u 10;y;x_2/y;u 9;$y \cdot (x_2/y)$;$y \cdot u$;r_1 15;x_1;x_2;z 9;$y \cdot (x_2/y)$;$y \cdot u + r_1$;z 9;$y \cdot u$;r_1;z 54;$y \cdot (x_2/y)$;$(y \cdot u + r_1) + z$;r_2 14;$y \cdot u$;$r_1 + z$;r_2 69;$y \cdot u$;r_2;y 95;y;u }

The *qed* proofs both are checked by the *qed* software and automatically generate the TeX code to display the compact *qed* proofs as shown above.

Nelson evidently intended the *qed* project to a be step towards automated proof systems that would support mathematicians supplying proofs in completely formal format that would be both human checkable and machine verifiable.

6 Some quotations.

Nelson's philosophy of formalism does not mean that mathematics is removed from the everyday, ordinary real world. Indeed, he apparently finds his very formal mathematics to be part and parcel of the ordinary world, including personal relationships and even religious feeling. Some of this philosophy can be found in the following quotations, which are taken from both published and unpublished sources.

> Numbers are divine, the only true divinity, the source of all that is in the world, holy, to be worshiped and glorified. Such is the Pythagorean religion, and such is the origin of mathematics. This is the religion from which I am apostate. Nelson [17, p.1]

This paper [17] contains Nelson's strongest statements about his formalism.

> It must be exhilarating to the superbly skilled people restoring the Sistine Chapel to reveal the original work that lay under the smoke and grime of centuries. I felt exhilaration writing Chapter 2: for the first time I experienced mathematics without the obscuring layer of semantics.
>
> [...] I feel confident now that complete formalization of mathematics is not only feasible, but practical. The question remains: is it worthwhile? To me the answer is clearly yes. Nelson [23, pp.88-89]

The previous quote and the next are from Nelson's manuscript using *qed*.

> [...] In the not distant future there will be huge data banks of theorems with rapid search procedures to help mathematicians construct proofs of new theorems. [...]
>
> But for centuries to come, human mathematicians will not be replaced by computers. We have different search skills. There is a phase transition separating feasible searches from infeasible ones, a phase transition that is roughly described by the distinction between polynomial time algorithms and exponential time algorithms. The latter are in general infeasible; they will remain forever beyond the reach of both people and machines. Nelson [23, p.89]

This explains what Nelson's expects for computer-based mathematics research, namely that for the foreseeable future (centuries) computers will not attain all the capabilities of humans. Nelson discusses this further in [19].

> Mathematicians no more *discover* truths than the sculptor discovers the sculpture inside the stone. (Surely you are joking, Mr. Buonarroti!) But unlike sculpting, our work is tightly constrained, both by the strict requirements of syntax and by the collegial nature of the enterprise. This is how mathematics differs profoundly from art. Nelson [18, p.4]

This indicates, in part, how Nelson reconciles pure formalism with the stability of the truth of statements like the odd perfect number question. It was also the inspiration for the "architect" analogy presented earlier in the present article.

I cannot resist ending with one more quote of Nelson's:

> Perhaps infinity is not far off in space or time or thought; perhaps it is while engaged in an ordinary activity — writing a page, getting a child ready for school, talking with someone, teaching a class, making love — that we are immersed in infinity. Nelson [22, p.50]

This quote was made during the formalization of predicative arithmetic, where the idea of mathematical infinite was being explored from Nelson's predicative viewpoint. It is clear that Nelson does not see any dichotomy between the mathematical and non-mathematical parts of life.

A Addition and multiplication as relations

This section describes how predicative arithmetic can be modified so as to remove the assumption that addition and multiplication are total functions.

We define a new base theory Q^- to replace the theory Q. The non-logical symbols of Q^- are the unary successor function S and the ternary relation symbols A and M. The intended meaning of the latter two symbols are that

$$A(n,m,p) \qquad \text{means} \qquad n+m=p,$$

and

$$M(n,m,p) \qquad \text{means} \qquad n\cdot m=p.$$

These two symbols are intended to replace the function symbols $+$ and $\cdot$ of Q.

The axioms of Q^- are as follows:

$$\begin{array}{ll}
Sx \neq 0 & A(x,0,x) \\
Sx = Sy \to x = y & A(x,y,z) \to A(x,Sy,Sz) \\
x \neq 0 \to (\exists y)(Sy = x) & M(x,0,0) \\
A(x,y,z) \wedge A(x,y,u) \to z = u & M(x,y,z) \wedge A(z,x,u) \to M(x,Sy,u) \\
M(x,y,z) \wedge M(x,y,u) \to z = u &
\end{array}$$

Note these are the direct translations of the axioms of Q plus axioms stating that A and M are not multi-valued. We define $\leq$ in Q^- by $x \leq y \leftrightarrow \exists z A(x,z,y)$. It is straightforward to show that Q is interpretable in Q^- via inductive cuts.

B Predicativity of exponential time

This section sketches a proof that there is a bounded theory T which is interpretable in Q via inductive cuts, and such that all exponential time functions are intensionally defined in T. We give mostly a hint of the proof. Recall that Chandra-Kozen-Stockmeyer [5] characterized exponential time computability in terms of alternating polynomial space computability. We restrict attention to exponential time functions f with polynomial growth rate (for more general exponential time functions, one would have to make do with the bit graph of the function). By Chandra-Kozen-Stockmeyer, the bit graph of the function f is computed by an alternating Turing machine which uses space $O(n^k)$ on input of length n, for some fixed k. From this, it is straightforward to construct functions g, h, r and s which are polynomial time computable so that

$$\begin{aligned}
f(x) &= h(0,0,x) \\
h(t,m,x) &= g(h(t+1,r(m,x),x),h(t+1,s(m,x),x)).
\end{aligned}$$

where there is an integer k so that $|h(t,m,x)| \leq |x|^k$ for all x and such that $h(t,m,x)=0$ whenever $|t| > |x|^k$ or $|m| > |x|^k$. The intuition is that t is a time parameter, that m serves as an instantaneous description of a step in the computation of $f(x)$, and that g, r and s implement a finite state controller.

We now form a predicative theory as follows. First define an inductive cut Q_3 which is closed under the $\#$ function and where bounded induction holds. On this cut, we say a number x is *small* if there is a $z = 2^x \in Q_3$. It is straightforward to introduce functions f, g and h so that the above two equations hold for all small x. The set of small elements forms an inductive cut closed under $\cdot$; by using the construction of Theorem 1, we find a subset of Q_3 which is also an inductive cut and is closed under $\#$ again. On this cut, the functions $f(x)$, $g(a,b)$ and $h(t,m,x)$ are all total and satisfy the two defining equations for f.

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