Threshold logic proof systems

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In this note, we show the intersimulation of three threshold logics within a polynomial size and constant depth factor. The logics are PTK, PTK' and FC, the latter introduced by J. Krajíček in [2].

Definition 1 *Propositional threshold logic* is given as follows. Formula depth and size are defined inductively by:

i. a propositional variable x_i, $i \in \mathbf{N}$, is a formula of depth 0 and size 1.[1]

ii. if F is a formula then $\neg F$ is a formula of depth $1 + dp(F)$ and size $1 + size(F)$.

iii. if $F_1, \ldots, F_n$ are formulas and $1 \leq k \leq n$ then $T_k^n(F_1, \ldots, F_n)$ is a formula of depth $1 + max\{depth(F_i) : 1 \leq i \leq n\}$ and size $(n + k) + 1 + \sum_{1 \leq i \leq n} size(F_i)$.

Propositional threshold logic can be viewed as an extension of propositional logic in the connectives $\neg, \wedge, \vee$, the latter two connectives being defined by

$$\bigvee_{1 \leq i \leq n} F_i \equiv T_1^n(F_1, \ldots, F_n)$$

$$\bigwedge_{1 \leq i \leq n} F_i \equiv T_n^n(F_1, \ldots, F_n)$$

A *cedent* is any sequence $F_1, \ldots, F_n$ of formulas separated by commas. Cedents are sometimes designated by $\Gamma, \Delta, \ldots$ (capital Greek letters). A *sequent* is given by $\Gamma \vdash \Delta$, where Γ, Δ are arbitrary cedents. The size [resp. depth] of a cedent $F_1, \ldots, F_n$ is $\sum_{1 \leq i \leq n} size(F_i)$ [resp. $\max_{1 \leq i \leq n}(depth(F_i))$]. The size [resp. depth] of a sequent $\Gamma \vdash \Delta$ is $size(\Gamma) + size(\Delta)$ [resp. $\max(depth(\Gamma), depth(\Delta))$]. The intended interpretation of the sequent $\Gamma \vdash \Delta$ is $\wedge\Gamma \rightarrow \vee\Delta$.

An *initial sequent* is of the form $F \vdash F$ where F is any formula of propositional threshold logic. The rules of inference of PTK, the sequent calculus of

[1] One could as well allow propositional constants 1 (TRUE) and 0 (FALSE) of depth 0 and size 1.

propositional threshold logic, are as follows.[2] By convention, $T^n_m(A_1, \ldots, A_n)$ is only defined if $1 \le m \le n$.

structural rules

$$\text{weak left: } \frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, A, \Delta \vdash \Gamma'} \qquad \text{weak right: } \frac{\Gamma \vdash \Gamma', \Delta'}{\Gamma \vdash \Gamma', A, \Delta'}$$

$$\text{contract left: } \frac{\Gamma, A, A, \Delta \vdash \Gamma'}{\Gamma, A, \Delta \vdash \Gamma'} \qquad \text{contract right: } \frac{\Gamma \vdash \Gamma', A, A, \Delta'}{\Gamma \vdash \Gamma', A, \Delta'}$$

$$\text{permute left: } \frac{\Gamma, A, B, \Delta \vdash \Gamma'}{\Gamma, B, A, \Delta \vdash \Gamma'} \qquad \text{permute right: } \frac{\Gamma \vdash \Gamma', A, B, \Delta'}{\Gamma \vdash \Gamma', B, A, \Delta'}$$

cut rule

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

logical rules

$$\neg\text{-left: } \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \qquad \neg\text{-right: } \frac{\Gamma \vdash A, \Delta}{\neg A, \Gamma \vdash \Delta}$$

$$\wedge\text{-left: } \frac{A_1, \ldots, A_n, \Gamma \vdash \Delta}{T^n_n(A_1, \ldots, A_n), \Gamma \vdash \Delta} \quad \text{for } n \ge 1$$

$$\wedge\text{-right: } \frac{\Gamma \vdash A_1, \Delta \qquad \cdots \qquad \Gamma \vdash A_n, \Delta}{\Gamma \vdash T^n_n(A_1, \ldots, A_n), \Delta} \quad \text{for } n \ge 1$$

$$\vee\text{-left: } \frac{A_1, \Gamma \vdash \Delta \qquad \cdots \qquad A_n, \Gamma \vdash \Delta}{T^n_1(A_1, \ldots A_n), \Gamma \vdash \Delta} \quad \text{for } n \ge 1$$

$$\vee\text{-right: } \frac{\Gamma \vdash A_1, \ldots, A_n, \Delta}{\Gamma \vdash T^n_1(A_1, \ldots, A_n), \Delta} \quad \text{for } n \ge 1$$

[2] Gentzen's original sequent calculus for first order logic was called *LK* (*Logischer Kalkül*). The propositional sequent calculus with connectives ¬, ∨, ∧ has sometimes been called *PK* (propositional Kalkül), so our *propositional threshold Kalkül* is denoted *PTK*.

$$T^n_k\text{-left:}\quad \frac{T^{n-1}_k(A_2,\ldots,A_n),\Gamma\vdash\Delta \qquad A_1,T^{n-1}_{k-1}(A_2,\ldots,A_n),\Gamma\vdash\Delta}{T^n_k(A_1,\ldots,A_n),\Gamma\vdash\Delta}\quad \text{for } 2\le k<n$$

$$T^n_k\text{-right:}\quad \frac{\Gamma\vdash A_1,T^{n-1}_k(A_2,\ldots,A_n),\Delta \qquad \Gamma\vdash T^{n-1}_{k-1}(A_2,\ldots,A_n),\Delta}{\Gamma\vdash T^n_k(A_1,\ldots,A_n),\Delta}\quad \text{for } 2\le k<n$$

The structural rules, cut rule, $\neg$ rules, $\wedge$ rules and $\vee$ rules are the same as for PTK. However, in place of the T^n_k rules of PTK, PTK' has the following rules.

$$T^n_k\text{-left1:}\quad \frac{T^n_k(A_1,\ldots,A_n),\Gamma\vdash\Delta}{T^n_{k+\ell}(A_1,\ldots,A_n),\Gamma\vdash\Delta}\quad \text{for } 1\le k<k+\ell\le n$$

$$T^n_k\text{-left2:}\quad \frac{T^n_k(A_1,\ldots,A_n),\Gamma\vdash\Delta}{T^{n+m}_{k+m}(A_1,\ldots,A_n,B_1,\ldots,B_m),\Gamma\vdash\Delta}\quad \text{for } 1\le k\le n<n+m$$

$$T^n_k\text{-left3:}\quad \frac{\neg A_1,\ldots,\neg A_n,T^m_k(B_1,\ldots,B_m),\Gamma\vdash\Delta}{\neg A_1,\ldots,\neg A_n,T^{m+n}_k(A_1,\ldots,A_n,B_1,\ldots,B_m),\Gamma\vdash\Delta}\quad \text{for } 1\le k\le m<m+n$$

$$T^n_k\text{-right1:}\quad \frac{\Gamma\vdash T^n_k(A_1,\ldots,A_n),\Delta}{\Gamma\vdash T^{n+m}_k(A_1,\ldots,A_n,B_1,\ldots,B_m),\Delta}\quad \text{for } 1\le k\le n<n+m$$

$$T^n_k\text{-right:}\quad \frac{\Gamma\vdash T^n_k(A_1,\ldots,A_n),\Delta \qquad \Gamma\vdash T^m_\ell(B_1,\ldots,B_m),\Delta}{\Gamma\vdash T^{n+m}_{k+\ell}(A_1,\ldots,A_n,B_1,\ldots,B_m),\Delta}\quad \text{for } 1\le k\le m<m+n$$

In [2], J. Krajíček introduced an extension of the Frege system F, called FC for Frege with counting. In addition to the usual connectives of F, *counting* connectives $C_{n,k}(x_1,\ldots,x_n)$ are admitted, whose interpretation is that exactly k of the x_i equal 1.

Definition 2 FC is the propositional proof system having connectives $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$ together with infinitely many new connectives $C_{n,k}(\phi_1,\ldots,\phi_n)$, for $1\le n$ and $k\le n$. The axioms of FC are those of F (see [1]) together with the new axioms:

1. $A\equiv C_{1,1}(A)$
2. $C_{n,0}(A_1,\ldots,A_n)\equiv(\neg A_1\wedge\ldots\wedge\neg A_n)$

3. $C_{n+1,k+1}(A_1,\ldots,A_{n+1}) \equiv$

$$\equiv [\ (C_{n,k}(A_1,\ldots,A_n) \wedge A_{n+1}) \ \vee \ (C_{n,k+1}(A_1,\ldots,A_n) \wedge \neg A_{n+1})\ ]$$

if $k < n$

4. $C_{n+1,n+1}(A_1,\ldots,A_{n+1}) \equiv [\ (C_{n,n}(A_1,\ldots,A_n) \wedge A_{n+1})\ ]$.

We intend to show the relation between FC and our threshold proof systems PTK and PTK'; namely that constant depth polynomial size FC proofs correspond to polynomial size constant depth PTK and PTK' proofs, and vice versa. We begin by simulating FC within PTK'.

Definition 3 Translate the FC formula A by the PTK' formula $\overline{A}$ as follows:

FC formula	PTK' formula
x	x
$\bigwedge_{i=1}^n A_i$	$T_n^n(\overline{A_1},\ldots,\overline{A_n})$
$\bigvee_{i=1}^n A_i$	$T_1^n(\overline{A_1},\ldots,\overline{A_n})$
$A \supset B$	$T_1^2(\neg\overline{A},\overline{B})$
$A \equiv B$	$T_2^2(\overline{A \supset B},\overline{B \supset A})$
$C_{n,k}(A_1,\ldots,A_n), 0 < k < n$	$T_2^2(T_k^n(\overline{A_1},\ldots,\overline{A_n}),\neg T_{k+1}^n(\overline{A_1},\ldots,\overline{A_n}))$
$C_{n,n}(A_1,\ldots,A_n)$	$T_n^n(\overline{A_1},\ldots,\overline{A_n})$
$C_{n,0}(A_1,\ldots,A_n)$	$\neg T_1^n(\overline{A_1},\ldots,\overline{A_n})$

For each axiom scheme A of FC, we sketch the PTK' proof of $\overline{A}$ (usually the last few steps from the formula $\tilde{A}$ proved to the equivalent $\overline{A}$ are easy and left to the reader). In our notation, $C_{n,k}(\vec{A})$ abbreviates $C_{n,k}(A_1,\ldots,A_n)$, and $T_k^n(\vec{A})$ abbreviates $T_k^n(A_1,\ldots,A_n)$. We often abbreviate A_{n+1} by A, so that for instance in the first subclaim appearing in the proof of Axiom 3 below,

$$T_{k+1}^{n+1}(\vec{A}) \vdash T_k^n(\vec{A}) \wedge A, T_{k+1}^n(\vec{A}) \wedge \neg A$$

abbreviates

$$T_{k+1}^{n+1}(A_1,\ldots,A_{n+1}) \vdash T_k^n(A_1,\ldots,A_n) \wedge A_{n+1}, T_{k+1}^n(A_1,\ldots,A_n) \wedge \neg A_{n+1}$$

Axiom 1 $x \equiv C_{1,1}(x)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{x \vdash x}{x \vdash T_1^1(x)}}{\vdash \neg x, T_1^1(x)}}{\vdash \neg x \vee T_1^1(x)} \qquad \dfrac{\dfrac{\dfrac{x \vdash x}{T_1^1(x) \vdash x}}{\vdash \neg T_1^1(x), x}}{\vdash \neg T_1^1(x) \vee x}}{\vdash (\neg x \vee T_1^1(x)) \wedge (\neg T_1^1(x) \vee x)}
$$

This completes the proof of axiom 1.

<u>Axiom 2</u> $C_{n,0}(A_1, \ldots, A_n) \equiv \neg A_1 \wedge \ldots \wedge \neg A_n$ (Recall that $\wedge, \vee$ associate to the left.)

<u>Claim</u> $\overline{C_{n,0}(A,B,C)} \vdash \overline{(\neg A \wedge \neg B) \wedge \neg C}$
<u>Pf</u>

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash T_1^1(A)}}{A \vdash T_1^3(A,B,C)}}{\neg T_1^3(A,B,C) \vdash \neg A} \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{B \vdash T_1^1(B)}}{B \vdash T_1^3(A,B,C)}}{\neg T_1^3(A,B,C) \vdash \neg B}}{\neg T_1^3(A,B,C) \vdash (\neg A \wedge \neg B)} \qquad \dfrac{\dfrac{\dfrac{C \vdash C}{C \vdash T_1^1(C)}}{C \vdash T_1^3(A,B,C)}}{\neg T_1^3(A,B,C) \vdash \neg C}}{\neg T_1^3(A,B,C) \vdash (\neg A \wedge \neg B) \wedge \neg C}
$$

<u>Claim</u> $\overline{(\neg A \wedge \neg B) \wedge \neg C} \vdash C_{3,0}(A,B,C)$
<u>Pf</u>

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \neg A \vdash}}{A, \neg A, \neg B \vdash}}{A, (\neg A \wedge \neg B) \vdash}}{A, (\neg A \wedge \neg B), \neg C \vdash}}{A, (\neg A \wedge \neg B \wedge \neg C) \vdash} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{B, \neg B \vdash}}{B, \neg A, \neg B \vdash}}{B, (\neg A \wedge \neg B) \vdash}}{B, (\neg A \wedge \neg B), \neg C \vdash}}{B, (\neg A \wedge \neg B \wedge \neg C) \vdash} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \vdash C}{C, \neg C \vdash}}{C, \neg A, \neg B, \neg C \vdash}}{C, (\neg A \wedge \neg B), \neg C \vdash}}{C, (\neg A \wedge \neg B \wedge \neg C) \vdash}}{T_1^3(A,B,C), (\neg A \wedge \neg B \wedge \neg C) \vdash}}{\neg A \wedge \neg B \wedge \neg C \vdash \neg T_1^3(A,B,C)}
$$

This completes the proof of axiom 2.

<u>Axiom 3</u> $C_{n+1,k+1}(\vec{A}) \equiv (C_{n,k}(\vec{A}) \wedge A_{n+1}) \vee (C_{n,k+1}(\vec{A}, A_{n+1}) \wedge \neg A_{n+1}$

<u>Claim</u> PTK' proves

$$
\overline{C_{n+1,k+1}}(\vec{A}) \vdash \overline{(C_{n,k}(\vec{A}) \wedge A_{n+1}) \vee (C_{n,k+1}(\vec{A}, A_{n+1}) \wedge \neg A_{n+1}}
$$

The claim follows from two subclaims.

Subclaim $T^{n+1}_{k+1}(\vec{A}) \vdash T^n_k(\vec{A}) \wedge A, T^n_{k+1}(\vec{A}) \wedge \neg A$
Pf

$$
\begin{prooftree}
\AxiomC{$T^n_{k+1}(\vec{A}) \vdash T^n_{k+1}(\vec{A})$}
\UnaryInfC{$\neg A, T^n_{k+1}(\vec{A}) \vdash T^n_{k+1}(\vec{A})$}
\UnaryInfC{$\neg A, T^{n+1}_{k+1}(\vec{A}) \vdash T^n_{k+1}(\vec{A})$}
\UnaryInfC{$T^{n+1}_{k+1}(\vec{A}) \vdash T^n_{k+1}(\vec{A}), \neg\neg A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$\vdash A, \neg A$}
\UnaryInfC{$\neg\neg A \vdash A$}
\BinaryInfC{$T^{n+1}_{k+1}(\vec{A}) \vdash A, T^n_{k+1}(\vec{A})$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$\vdash A, \neg A$}
\UnaryInfC{$T^{n+1}_{k+1}(\vec{A}) \vdash A, \neg A$}
\BinaryInfC{$T^{n+1}_{k+1}(\vec{A}) \vdash A, T^n_{k+1}(\vec{A}) \wedge \neg A$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$T^n_k(\vec{A}) \vdash T^n_k(\vec{A})$}
\UnaryInfC{$T^{n+1}_{k+1}(\vec{A}) \vdash T^n_k(\vec{A})$}
\UnaryInfC{$T^{n+1}_{k+1}(\vec{A}) \vdash T^n_{k+1}(\vec{A}) \wedge \neg A, T^n_k(\vec{A})$}
\end{prooftree}
$$

Combining the last lines of the previous two proofs using $\wedge$-right, we have

$$T^{n+1}_{k+1}(\vec{A}) \vdash T^n_k(\vec{A}) \wedge A, T^n_{k+1}(\vec{A}) \wedge \neg A$$

which establishes the subclaim.

Subclaim $\neg T^{n+1}_{k+2}(\vec{A}) \vdash \neg T^n_{k+2}(\vec{A}) \wedge \neg A, \neg T^n_{k+1} \wedge A$
Pf First we prove the following.

$$
\begin{prooftree}
\AxiomC{$T^n_{k+2}(\vec{A}) \vdash T^n_{k+2}(\vec{A})$}
\UnaryInfC{$T^n_{k+2}(\vec{A}) \vdash T^{n+1}_{k+2}(\vec{A})$}
\UnaryInfC{$T^n_{k+2}(\vec{A}) \vdash T^{n+1}_{k+2}(\vec{A}), \neg T^n_{k+1}$}
\UnaryInfC{$\neg T^{n+1}_{k+2}(\vec{A}) \vdash \neg T^n_{k+2}(\vec{A}), \neg T^n_{k+1}(\vec{A})$}
\AxiomC{$T^n_{k+2}(\vec{A}) \vdash T^n_{k+2}(\vec{A})$}
\UnaryInfC{$T^n_{k+2}(\vec{A}) \vdash T^{n+1}_{k+2}(\vec{A})$}
\UnaryInfC{$\neg T^{n+1}_{k+2}(\vec{A}) \vdash \neg T^n_{k+2}(\vec{A})$}
\UnaryInfC{$\neg T^{n+1}_{k+2}(\vec{A}) \vdash \neg T^n_{k+2}(\vec{A}), A$}
\BinaryInfC{$\neg T^{n+1}_{k+2}(\vec{A}) \vdash \neg T^n_{k+2}(\vec{A}), \neg T^n_{k+1}(\vec{A}), A$}
\end{prooftree}
$$

Second we prove the following.

$$
\begin{prooftree}
\AxiomC{$T^n_{k+1}(\vec{A}) \vdash T^n_{k+1}(\vec{A})$}
\UnaryInfC{$A, T^n_{k+1}(\vec{A}) \vdash T^n_{k+1}(\vec{A})$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$A, T^n_{k+1}(\vec{A}) \vdash T^1_1(A)$}
\BinaryInfC{$A, T^n_{k+1}(\vec{A}) \vdash T^{n+1}_{k+2}(\vec{A})$}
\UnaryInfC{$A, \neg T^{n+1}_{k+2}(\vec{A}) \vdash \neg T^n_{k+1}(\vec{A})$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$A, \neg T^{n+1}_{k+2} \vdash A$}
\BinaryInfC{$A, \neg T^{n+1}_{k+2}(\vec{A}) \vdash \neg T^n_{k+1}(\vec{A}) \wedge A$}
\UnaryInfC{$\neg T^{n+1}_{k+2}(\vec{A}) \vdash \neg A, \neg T^n_{k+1}(\vec{A}) \wedge A$}
\end{prooftree}
$$

Combining the last lines of the previous two proofs using $\wedge$-right, we have

$$\neg T_{k+2}^{n+1}(\vec{A}) \vdash \neg T_{k+2}^{n}(\vec{A}) \wedge \neg A, \neg T_{k+1}^{n}(\vec{A}) \wedge A$$

as desired. Now from both subclaims, it can be shown that

$$T_{k+1}^{n+1}(\vec{A}) \wedge \neg T_{k+2}^{n+1}(\vec{A}) \vdash T_{k}^{n}(\vec{A}) \wedge \neg T_{k+1}^{n}(\vec{A}) \wedge A, T_{k+1}^{n}(\vec{A}) \wedge \neg T_{k+2}^{n}(\vec{A}) \wedge \neg A.$$

This establishes the claim that

$$\overline{C_{n+1,k+1}(\vec{A})} \vdash \overline{(C_{n,k}(\vec{A}) \wedge A_{n+1}) \vee (C_{n,k+1}(\vec{A}) \wedge \neg A_{n+1})}$$

<u>Claim</u> PTK' proves the converse of the previous, i.e.

$$\overline{(C_{n,k}(\vec{A}) \wedge A_{n+1}) \vee (C_{n,k+1}(\vec{A}) \wedge \neg A_{n+1})} \vdash \overline{C_{n+1,k+1}(\vec{A})}$$

This translates to

$$(T_{k}^{n}(\vec{A}) \wedge \neg T_{k+1}^{n}(\vec{A}) \wedge A) \vee (T_{k+1}^{n}(\vec{A}) \wedge T_{k+2}^{n}(\vec{A}) \wedge \neg A) \vdash T_{k+1}^{n+1}(\vec{A}) \wedge \neg T_{k+2}^{n+1}(\vec{A}).$$

The claim follows from two subclaims.
<u>Subclaim</u> $(T_{k}^{n}(\vec{A}) \wedge \neg T_{k+1}^{n}(\vec{A}) \wedge A) \vee (T_{k+1}^{n}(\vec{A}) \wedge T_{k+2}^{n}(\vec{A}) \wedge \neg A) \vdash T_{k+1}^{n+1}(\vec{A})$
<u>Pf</u>

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash T_1^1(A)}}{A, T_k^n(\vec{A}) \vdash T_1^1(A)} \qquad \dfrac{T_k^n(\vec{A}) \vdash T_k^n(\vec{A})}{A, T_k^n(\vec{A}) \vdash T_k^n(\vec{A})}}{A, T_k^n(\vec{A}) \vdash T_{k+1}^{n+1}(\vec{A})}}{A, T_k^n(\vec{A}), \neg T_{k+1}^n(\vec{A}) \vdash T_{k+1}^{n+1}(\vec{A})}$$

$$\dfrac{\dfrac{\dfrac{T_{k+1}^n(\vec{A}) \vdash T_{k+1}^n(\vec{A})}{T_{k+1}^n(\vec{A}) \vdash T_{k+1}^{n+1}(\vec{A})}}{\neg A, T_{k+1}^n(\vec{A}) \vdash T_{k+1}^{n+1}(\vec{A})}}{\neg A, T_{k+1}^n(\vec{A}), \neg T_{k+2}^n(\vec{A}) \vdash T_{k+1}^{n+1}(\vec{A})}$$

Now combining the last two proofs using $\vee$-left, we have

$$(A, T_k^n(\vec{A}), \neg T_{k+1}^n(\vec{A})) \vee (\neg A, T_{k+1}^n(\vec{A}), \neg T_{k+2}^n(\vec{A})) \vdash T_{k+1}^{n+1}(\vec{A})$$

<u>Subclaim</u> $(T_{k}^{n}(\vec{A}) \wedge \neg T_{k+1}^{n}(\vec{A}) \wedge A) \vee (T_{k+1}^{n}(\vec{A}) \wedge T_{k+2}^{n}(\vec{A}) \wedge \neg A) \vdash \neg T_{k+2}^{n+1}(\vec{A})$
<u>Pf</u>

$$\dfrac{\dfrac{\dfrac{\dfrac{T^n_{k+1}(\vec{A}) \vdash T^n_{k+1}(\vec{A})}{T^{n+1}_{k+2}(\vec{A}) \vdash T^n_{k+1}(\vec{A})}}{\neg T^n_{k+1}(\vec{A}) \vdash \neg T^{n+1}_{k+2}(\vec{A})}}{A, T^n_k(\vec{A}), \neg T^n_{k+1}(\vec{A}) \vdash \neg T^{n+1}_{k+2}(\vec{A})} \qquad \dfrac{\dfrac{\dfrac{\dfrac{T^n_{k+2}(\vec{A}) \vdash T^n_{k+2}(\vec{A})}{\neg A, T^n_{k+2}(\vec{A}) \vdash T^n_{k+2}(\vec{A})}}{\neg A, T^{n+1}_{k+1}(\vec{A}) \vdash T^n_{k+2}(\vec{A})}}{\neg A, T^n_{k+2}(\vec{A}) \vdash \neg T^{n+1}_{k+2}(\vec{A})}}{\neg A, T^n_{k+1}(\vec{A}), \neg T^n_{k+2}(\vec{A}) \vdash \neg T^{n+1}_{k+2}(\vec{A})}}{(A, T^n_k(\vec{A}), \neg T^n_{k+1}(\vec{A})) \vee (\neg A, T^n_{k+1}(\vec{A}), \neg T^n_{k+2}(\vec{A})) \vdash \neg T^{n+1}_{k+2}(\vec{A})}$$

From the two subclaims, we obtain a proof of

$$(T^n_k(\vec{A}) \wedge \neg T^n_{k+1}(\vec{A}) \wedge A) \vee (T^n_{k+1}(\vec{A}) \wedge T^n_{k+2}(\vec{A}) \wedge \neg A) \vdash T^{n+1}_{k+1}(\vec{A}) \wedge \neg T^{n+1}_{k+2}(\vec{A})$$

which establishes

$$\overline{(C_{n,k}(\vec{A}) \wedge A) \vee (C_{n,k+1}(\vec{A}) \wedge \neg A)} \vdash \overline{C_{n,k+1}(\vec{A})}.$$

This concludes the proof of axiom 3.

<u>Axiom 4</u> $C_{n+1,n+1}(\vec{A}) \equiv C_{n,n}(\vec{A}) \wedge A$

<u>Claim</u> $\overline{C_{n+1,n+1}(\vec{A})} \vdash \overline{C_{n,n}(\vec{A})} \wedge A$.
<u>Pf</u> Show $T^{n+1}_{n+1}(\vec{A}) \vdash T^n_n(\vec{A}) \wedge A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A_1 \vdash A_1}{A_1, \ldots, A_{n+1} \vdash A_1}}{T^{n+1}_{n+1}(\vec{A}) \vdash A_1} \qquad \dfrac{\dfrac{A_n \vdash A_n}{A_1, \ldots, A_{n+1} \vdash A_n}}{T^{n+1}_{n+1}(\vec{A}) \vdash A_n} \qquad \dfrac{\dfrac{A_{n+1} \vdash A_{n+1}}{A_1, \ldots, A_{n+1} \vdash A_{n+1}}}{T^{n+1}_{n+1}(\vec{A}) \vdash A_{n+1}}}{T^{n+1}_{n+1}(\vec{A}) \vdash T^n_n(\vec{A})}}{T^{n+1}_{n+1}(\vec{A}) \vdash T^n_n(\vec{A}) \wedge A_{n+1}}$$

This completes the proof of the claim.

<u>Claim</u> $\overline{C_{n,n}(\vec{A}) \wedge A} \vdash \overline{C_{n+1,n+1}(\vec{A})}$
<u>Pf</u> Show $T^n_n(\vec{A}) \wedge A \vdash T^{n+1}_{n+1}(\vec{A})$.

$$\dfrac{\dfrac{\dfrac{A_1 \vdash A_1}{A_1, \ldots, A_{n+1} \vdash A_1}}{T^n_n(\vec{A}), A_{n+1} \vdash A_1} \qquad \dfrac{\dfrac{A_n \vdash A_n}{A_1, \ldots, A_{n+1} \vdash A_n}}{T^n_n(\vec{A}), A_{n+1} \vdash A_n} \qquad \dfrac{\dfrac{A_{n+1} \vdash A_{n+1}}{A_1, \ldots, A_{n+1} \vdash A_{n+1}}}{T^n_n(\vec{A}), A_{n+1} \vdash A_{n+1}}}{T^n_n(\vec{A}), A_{n+1} \vdash T^{n+1}_{n+1}(\vec{A})}$$

This completes the proof of the claims and so establishes the provability of the translation of Axiom 4 in PTK'.

By *depth* and *size* of a proof in a propositional proof system such as F, FC, PTK', etc. we mean the maximum depth and size of any formula appearing in the proof (in particular, we do not mean the depth of the proof tree in a sequent calculus proof).

Theorem 4 *Suppose that $\langle P_n : n \geq 1 \rangle$ is a family of FC proofs, where P_n is a depth $d(n)$, size $s(n)$ proof of ϕ_n. Then there exists a constant c for which there exists a family $\langle P'_n : n \geq 1 \rangle$ of PTK' proofs, where P'_n is a depth $c + d(n)$, size $c \cdot s(n)$ proof of $\overline{\phi_n}$.*

Proof. The axioms of FC have previously been treated, and modus ponens (the only rule of inference of FC) is a special case of the cut rule of PTK'. Analysis of the previous PTK' proofs of the axioms of FC gives appropriate constant c. ∎

We now consider the simulation of PTK by FC.

Definition 5 Translate the PTK formula A by the FC formula $\tilde{A}$ as follows:

PTK formula	FC formula
x	x
$\neg A$	$\neg \tilde{A}$
$T^n_k(A_1, \ldots, A_n)$	$\bigvee_{i=k}^n C_{n,i}(\tilde{A}_1, \ldots, \tilde{A}_n)$

A PTK sequent $\Gamma \vdash \Delta$, which is equivalent to the formula

$$\bigwedge_{i=1}^{n} A_i \supset \bigvee_{j=1}^{m} B_j$$

is translated by the FC formula

$$\bigwedge_{i=1}^{n} \tilde{A}_i \supset \bigvee_{j=1}^{m} \tilde{B}_j.$$

Theorem 6 *Suppose that $\langle P_n : n \geq 1 \rangle$ is a family of PTK proofs, where P_n is a depth $d(n)$, size $s(n)$ proof of ϕ_n. Then there exists a constant c for which there exists a family $\langle P'_n : n \geq 1 \rangle$ of FC proofs, where P'_n is a depth $c + d(n)$, size $s(n)^c$ proof of $\tilde{\phi}_n$.*

Proof sketch By induction on the number of proof inferences. For each axiom of PTK, the translation of its sequent is easily provable in FC. Similarly, an appropriate translation of each proof rule of PTK is provable in FC. For instance, a binary rule

$$\frac{A_1, \ldots, A_{n_1} \vdash B_1, \ldots, B_{n_2} \qquad C_1, \ldots, C_{n_3} \vdash D_1, \ldots, D_{n_4}}{E_1, \ldots, E_{n_5} \vdash F_1, \ldots, F_{n_6}}$$

is translated into

$$(\bigwedge_{i=1}^{n_1} \tilde{A}_i \supset \bigvee_{i=1}^{n_2} \tilde{B}_i) \wedge (\bigwedge_{i=1}^{n_3} \tilde{C}_i \supset \bigvee_{i=1}^{n_4} \tilde{D}_i)$$

$$\supset$$

$$(\bigwedge_{i=1}^{n_5} \tilde{E}_i \supset \bigvee_{i=1}^{n_6} \tilde{F}_i)$$

To prove in FC the translation of the rule T_k^n-left, begin with the tautology

$$(\bigvee_{k \leq i \leq n} C_{n,i} \wedge \Gamma \supset \Delta) \supset (\bigvee_{k \leq i \leq n} C_{n,i} \wedge \Gamma) \supset \Delta$$

Using an axiom of FC, obtain

$$\bigvee_{k \leq i \leq n} ((A \wedge C_{n-1,i-1}) \vee (\neg A \wedge C_{n-1,i})) \wedge \Gamma \supset \Delta \cdots$$

This is equivalent to the following.

$$((A \wedge \bigvee_{k \leq j < n} C_{n-1,j}) \vee (\neg A \wedge \bigvee_{k \leq i < n} C_{n-1,i})) \wedge \Gamma \supset \Delta \cdots$$

Using the translation into FC of T_k^n (and for notational simplicity denoting the translation of formulas A by themselves), this yields the following.

$$((A \wedge T_{k-1}^{n-1}) \vee (\neg A \wedge T_k^{n-1})) \wedge \Gamma \supset \Delta \cdots$$

By distribution of $\wedge$ this yields

$$((A \wedge T_{k-1}^{n-1} \wedge \Gamma) \vee (\neg A \wedge T_k^{n-1} \wedge \Gamma)) \supset \Delta \cdots$$

By distribution of $\supset$ this yields

$$((A \wedge T_{k-1}^{n-1} \wedge \Gamma) \supset \Delta) \wedge ((\neg A \wedge T_k^{n-1} \wedge \Gamma) \supset \Delta)$$

$$\supset$$

$$(T_k^n \wedge \Gamma) \supset \Delta$$

It will be shown in the proof of the next theorem that

$$T_k^{n-1} \vdash T_{k-1}^{n-1}$$

and so

$$A \wedge T_k^{n-1} \wedge \Gamma \vdash A \wedge T_{k-1}^{n-1} \wedge \Gamma$$

From this, since

$$T_k^{n-1} \vdash (A \wedge T_k^{n-1}) \vee (\neg A \wedge T_k^{n-1})$$

it is not hard to see that there is an FC proof of the following.

$$((A \wedge T_{k-1}^{n-1} \wedge \Gamma) \supset \Delta) \wedge ((T_k^{n-1} \wedge \Gamma) \supset \Delta)$$

$$\supset$$

$$(T_k^n \wedge \Gamma) \supset \Delta$$

But this is the translation of rule T_k^n-left into FC. The FC proof of the translation of T_k^n-right is similar. ∎

Theorem 7 *Suppose that $\langle P_n : n \geq 1 \rangle$ is a family of PTK' proofs, where P_n is a depth $d(n)$, size $s(n)$ proof of ϕ_n. Then there exists a constant c for which there exists a family $\langle P_n' : n \geq 1 \rangle$ of PTK proofs, where P_n' is a depth $c + d(n)$, size $c \cdot s(n)$ proof of ϕ_n.*

Proof. Note first that

$$\dfrac{T_k^n \vdash T_k^n \qquad \dfrac{T_k^n \vdash T_{k-1}^n}{T_k^n \vdash A, T_{k-1}^n}}{T_k^n \vdash T_k^{n+1}}$$

and that

$$\dfrac{T_k^{n-1} \vdash T_{k-1}^{n-1} \qquad \dfrac{T_{k-1}^{n-1} \vdash T_{k-1}^n}{A, T_{k-1}^{n-1} \vdash T_{k-1}^n}}{T_k^n \vdash T_{k-1}^n}$$

Thus the $n \cdot k$ proof of $T_j^i \vdash T_j^{i+1}$ and $T_{j+1}^i \vdash T_j^i$ for $i < n$ and $j < k$ together yield a proof of

$$T_k^n \vdash T_k^{n+1} \tag{1}$$

Case 1: T_k^n-left1

Since T_k^n has size $O(n)$, there is an $n^{O(1)}$ size proof of (1). Now

$$\dfrac{T_{k+\ell}^n \vdash T_k^n \qquad T_k^n, \Gamma \vdash \Delta}{T_{k+\ell}^n, \Gamma \vdash \Delta}$$

Case 2: T_k^n-left2

$$\dfrac{T_{k+1}^n \vdash T_{k+1}^n \qquad A, T_k^n \vdash A \wedge T_k^n}{T_{k+1}^{n+1} \vdash A \wedge T_k^n, T_{k+1}^n}$$

From this, we obtain

$$T^{n+1}_{k+1} \vdash T^n_k$$

and by iteration rule T^n_k-left2.

Case 3: T^n_k-left3

$$\frac{T^n_{k+1}, \neg A \vdash \neg A \wedge T^n_{k+1} \qquad A, T^n_k, \neg A \vdash}{T^{n+1}_{k+1}, \neg A \vdash \neg A \wedge T^n_{k+1}}$$

by using the T^n_k-left rule of PTK. Iterating this, we have the proof of the T^n_k-left3 rule of PTK'.

Case 4: T^n_k-right1

Immediate from (1).

Case 5: T^n_k-right2

Iterating the idea of proof of case 2, we can show that

$$T^n_k(\vec{A}) \wedge T^m_\ell(\vec{B}) \vdash T^{n+m}_{k+\ell}(\vec{A}, \vec{B})$$

From this, case 5 follows.

This completes the proof of the theorem. ∎

It is not difficult to see that the simulations of FC, PTK and PTK' are within a polynomial factor of the size and a constant factor of the depth.

References

[1] S. A. Cook and R. Reckhow. On the relative efficiency of propositional proof systems. *Journal of Symbolic Logic*, 44:36 – 50, 1977.

[2] J. Krajíček. On Frege and extended Frege systems. In P. Clote and J. Remmel, editors, *Feasible Mathematics II*, pages 284—319. Birkhäuser, 1994.