

Ultrafinitism, Predicativism & Pragmatics

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Ultrafinitism: Physics, Mathematics, and Philosophy
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Talk outline:

- **Interpretability in Robinson's Q as “predicativity” (ultrafinism).**
 - A.S. Yessenin-Volpin ['61, '70, '81] in sense of Ed Nelson ['86].
 - Theories: $I\Delta_0$, Ω_1 -functions, S_2^i , $I\Delta_0 + O(1)$ -exponentials.
 - Computational Complexity: P, Pspace, Exptime.
 - Formalization of metamathematics and Gödel Incompleteness.
- **Logical, axiom-based characterizations of computational complexity**
 - S_2^1 , $T_2^0(\text{MSP})$, U_2^1 , PRA, PA, ...
- **What is the nature of mathematics?**
 - Two definitions

Part I: Predicative Arithmetic

(Acknowledgement: We largely follow E Nelson [1986], who based his Predicative Arithmetic on Yessenin-Volpin's work and Shoenfield's textbook as well as work of Gentzen, Parikh, Solovay, Paris, Wilkie, Pudlák, and others.)

Starting point: the axioms of Robinson's system Q and the Hilbert conception of integers in unary (or “tally”) representation. Basic, simplistic facts about 0, successor/predecessor (± 1), addition and *multiplication* are taken as given.

Syntax and formalization are taken as paramount. Semantics is treated with suspicion and viewed as less important.

There is no assumption that the integers exist as a completed infinity. However, there is an assumption of a universe of objects satisfying Q .

Base Language and Axioms of Predicative Arithmetic

Classical First-Order Logic: $\wedge, \vee, \neg, \rightarrow, \forall, \exists, =$

Function Symbols:

$0, S$ (successor), $+$ (addition), $\cdot$ (multiplication).

Robinson's Theory Q :

$$\begin{array}{ll} Sx \neq 0 & x + 0 = x \\ Sx = Sy \rightarrow x = y & x + Sy = S(x + y) \\ x \neq 0 \rightarrow (\exists y)(Sy = x) & x \cdot 0 = 0 \\ & x \cdot Sy = x \cdot y + x. \end{array}$$

Q is very weak; e.g., it does not prove $x + y = y + x$ or $0 + x = x$.
But Q is still subject to Gödel's incompleteness theorem.

Q does *not* include any induction axioms.

It is a very weak subtheory of Peano arithmetic (PA).

For convenience, we include associativity of $+$, $\cdot$ in Q for sake of discussion.

Extensions by Definition

The theory Q can be conservatively extended by adding new symbols defined in terms of old ones. For example:

Inequality Predicate:

$$x \leq y \quad \Leftrightarrow \quad (\exists z)(x + z = y)$$

Predecessor Function:

$$P(x) = y \quad \Leftrightarrow \quad S(y) = x \vee (x = 0 \wedge y = 0).$$

Extension by definitions form a conservative extension of Q , so can be used freely.

The consistency of assuming $0 + x = x$

Def'n A formula $\phi(x)$ is **inductive**, i.e., closed under successor, provided it has been proved that

$$\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(Sx)).$$

Theorem: (Solovay[unpub.], Nelson[1986], c.f. Gentzen)

Suppose $\phi(x)$ is inductive. Let

$$\phi^1(x) \text{ be } \forall y(y \leq x \rightarrow \phi(y)).$$

$$\phi^2(x) \text{ be } \forall y(\phi^1(y) \rightarrow \phi^1(y + x)).$$

$$\phi^3(x) \text{ be } \forall y(\phi^2(y) \rightarrow \phi^2(y \cdot x)).$$

Then, $\phi^3(x)$ defines an initial segment of the integers which is closed under S , $+$ and $\cdot$.

Thm: Let $\psi(x)$ be the formula $0 + x = x$. Then, $\psi(x)$ is inductive.

Pf: (a) $0 + 0 = 0$ holds by an axiom of Q .

(b) Suppose $0 + x = x$. Then $0 + Sx = S(0 + x) = Sx$. □.

Thus $\{x : \psi^3(x)\}$ is a set of integers that satisfies

$$Q_2 := Q + \forall x(0 + x = x).$$

This has provided an “interpretation” of Q_2 in Q .

Semantic viewpoint: starting with a mass of ‘integers’ that satisfy Q , we have found an initial segment of ‘integers’ that also satisfies $\forall x(0 + x = x)$.

Syntactic viewpoint: if Q is consistent, then Q_2 is consistent. Using ‘relativization’, any Q_2 -proof can be transformed into a Q -proof. By relativization is meant restricting attention to integers that satisfy $\psi^3(x)$.

Def'n [Nelson, 1986] A theory $T \supset Q$ is ***predicative*** if it is interpretable in Q . This include allows multiple uses of extension by definition and of interpretation with initial segments obtained from inductive formulas.

Examples of predicative principles:

1. Induction and the least number principle on bounded formulas. Bounded formulas may only use quantifiers which are bounded, $\forall x < t$ and $\exists x < t$.
3. Sequence coding, Gödel numbers of formulas and proofs. The smash function $\#$, $x\#y = 2^{|x| \cdot |y|}$, equivalently the Ω_1 -function $x^{\log y}$. (And, thus, substitution.)
4. Metamathematic concepts including consistency and interpretability.
5. The *best* proof of the Gödel incompleteness theorem. “Best” in that it uses “intensional” reasoning in the sense of Feferman, but starts with Q instead of, say, PRA or PA. [Pudlák, 1985]
6. Portions of real analysis [Ko-Friedman’82, Hook’83, Fernandes-Ferreira’14]

Some principles which are **not** predicative include:

1. The totality of exponentiation: $\text{exp} := \forall x \exists y (2^x = y)$.
2. Having an inductive initial segment on which superexponentiation $2 \uparrow x$ is total. Here, $2 \uparrow 0 = 1$ and $2 \uparrow (x + 1) = 2^{2 \uparrow x}$.
3. The Gentzen cut elimination theorem.
4. The consistency of the theory Q .

However, w.r.t. 1., principles that follow from a *finite* number of uses of exponentiation are predicative. E.g., the ***tautological consistency*** of Q and the ***bounded consistency*** of Q .

Furthermore, Paris and Wilkie showed that any bounded formula which is a consequence of $Q + \text{exp}$ is predicative. Also:

Thm: [Buss, '06] Polynomial space and exponential time functions that have polynomial growth rate are predicative as total functions. This apparently delimits the power of predicate theories.

Part II: Axiom-based definitions of complexity classes

Formal theories of arithmetic provide, in some cases, purely logical and axiom-based, definitions of computational complexity classes. The complexity classes can be predicative or non-predicative.

Desideratum: Definitions of complexity classes that avoid direct use of computational constructions; in particular, that use neither machine models nor functional (recursive) definitions of complexity classes.

Examples:

1. The bounded arithmetic theory S_2^1 uses length or polyomial induction on Σ_1^b -formulas: these express precisely properties in NP. It can define (in a nice way, Σ_1^b -definitions) exactly the polynomial predicates and functions. [Buss'85]
2. The bounded arithmetic theory $T_2^0(\text{MSP})$ uses usual induction on sharply bounded formulas (expressing a subclass of alternating log time). It defines (in the same way) exactly the polynomial predicates and functions. [Jeřábek, 2006]. $T_2^0(\text{MSP})$ is shown to be equivalent to PV_1 .
3. The second-order (predicative) theory U_2^1 defines exactly the polynomial space computable functions.

And some non-predicative examples:

4. PA (Peano arithmetic) defines exactly the ϵ_0 -recursive functions. [Gentzen]
5. $I\Sigma_1$ (induction on r.e. predicates) defines exactly the primitive recursive functions. [Parsons '70, Mints '71/2, Takeuti '74].

Open Questions:

- Can we find similar logical and axiom-based characterizations of alternating log time (uniform NC^1), or log space (L), or non-deterministic log space (NL)? The extant theories use axioms that state explicitly the totality of a many-one complete problem from the class. (C.f. Cook-Nguyen [2010].)
- The theory of least fixed points (LFP's) [Vardi 1982, Immerman 1984] gives a non-axiomatic logical characterization of polynomial time (P). Can one formulate an elegant first-order theory for P based on LFP's?

Part III: Definition of mathematics, #1

Mathematics is the study of objects and constructions, or of aspects of objects and constructions, which are capable of being fully and completely defined.

A defining characteristic of mathematics is that once mathematical objects are sufficiently well-specified then mathematical reasoning can be carried out with a robust and objective standard of rigor.

[Buss, FOM posting on NYC Logic Conference and Panel Discussion, Dec. 6, 1999.] See also critiques (and some support) from V. Sazonov, S. Simpson, H. Friedman, V. Pratt, and M. Insall in FOM discussions (Dec 1999-Jan 2000).

See also, Polya [1954]: “Numbers and figures are not the only objects of mathematics. Mathematics is basically inseparable from logic, and it deals with all objects which may be objects of an exact theory.”

Talking points for the first definition of mathematics.

1. This definition is a rejection of definitions of mathematics in terms of human activity, and the idea that mathematics should be defined as a social-historical phenomenon, not as an objectively definable field of study (e.g. Hersch [1999]).
2. Mathematical objects are generally only partially specified. E.g., sets do not have tangible properties such as “location” or “color”; rather sets are mental constructs that satisfy certain fully and completely defined properties. These aspects of sets are subject to mathematical study. (Similarly for integers or reals, etc.)
3. Objects of mathematical study do not need to be constructed or even uniquely specified. Correspondingly, this definition of mathematics is orthogonal to considerations of ultrafinitism.

Talking points continued ...

3. “Fully and completely defined” means that reasoning about these properties can be carried out with a “robust and objective standard of rigor”.
4. The rigor is essentially the kind of rigor that is seen in first-order logic. Furthermore, by Gödel’s completeness theorem, the kind of rigor that is seen in first-order logic is the ultimate level of rigor.
5. However, one does not need to use first-order logic to apply a “robust and objective standard of rigor” in mathematical reasoning. E.g. even before the advent of formal methods and first-order logic, people were doing mathematics!
6. As support for the notion of “robust and objective standard”, note that disagreements over the correctness of mathematical proofs can essentially always be resolved by breaking down reasoning steps into smaller and smaller units. Irreconcilable differences in what constitutes rigorous reasoning are very rare.

7. This definition of mathematics intentionally downplays the importance of set theory, as well as type theories, PA, or PRA, or predicative theories!
8. This definition of mathematics allows for a kind of platonism, but a rather unusual kind, where objects can have some aspects that are “fully and completely defined” and other aspects that are not.

It is an unusual situation, but it arises outside of mathematics too. For example, we have a conception of (mythical) unicorns that have some definite properties (e.g., are white and have a single horn), but lack other properties (e.g., what is the gestation period of a unicorn? Are they mortal?)

9. The definition is well-suited to “pure” mathematics, as well as “experimental” or “computational” mathematics.
It does not account well for the usefulness of mathematics in applications.
10. The definition should not be construed as implying that mathematics is superior to other fields. It is both a strength and a weakness of mathematics that it deals (only) with “fully and completely defined” aspects of objects.
11. Finally: This definition of mathematics is does not “fully and completely define” what is mathematics. It is not a mathematical definition, rather a philosophical definition.

Mathematics is a refined form of common sense.

This is not a **definition** per se, rather is a descriptive, or even prescriptive characterization of mathematics.

Indeed, essentially any scientific endeavor or other intellectual endeavor uses refined common sense (among other tools).

MOTIVATION: For guiding a philosophy of mathematics education, especially K-12.

Talking points for the second definition.

1. Mathematics education should expect students to be able integrate common sense reasoning with formal calculation. And to understand that formal calculation is itself a form of common sense reasoning.
2. Mathematics should not be taught as merely a set of tools or techniques to be applied in certain situations.
3. Students should expect mathematics to “make sense”.
4. Given that all students are expected to master common sense reasoning skills (e.g. doing a puzzle, preparing meals, planning a vacation, etc.), they should all be expected to master mathematical skills including abstract reasoning skills.
(Compare to Stanovich-West [2000], Khait [2004] and their System 1 versus System 2 reasoning.)

Open Problem

How might (or, should) the nature of mathematics change in the event that AI's are substantially better at mathematics than humans?

This seems to be more of a question of "When?" rather than "If ever?"

Thank you!