

# SUB-COMPUTABLE BOUNDED RANDOMNESS

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**ABSTRACT.** This paper defines a new notion of bounded computable randomness for certain classes of sub-computable functions which lack a universal machine. In particular, we define such versions of randomness for primitive recursive functions and for PSPACE functions. These new notions are robust in that there are equivalent formulations in terms of (1) Martin-Löf tests, (2) Kolmogorov complexity, and (3) martingales. We show these notions can be equivalently defined with prefix-free Kolmogorov complexity. We prove that one direction of van Lambalgen's theorem holds for relative computability, but the other direction fails. We discuss statistical properties of these notions of randomness.

## 1. INTRODUCTION

The study of algorithmic randomness has flourished over the past century. The main topic of study in this paper is the randomness of a single real number which, for our purposes, can be thought of as an infinite sequence  $X = (X(0), X(1), \dots)$  from  $\{0, 1\}^\omega$ . Many interesting notions of algorithmic randomness for real numbers have been investigated in recent years. The most well-studied notion, Martin-Löf randomness [27] or 1-randomness, is usually defined in terms of measure. Thus we say a real  $X$  is 1-random if it is *typical*, that is,  $X$  does not belong to any effective set of measure zero in the sense of Martin-Löf [27]. A second definition of 1-randomness may be given in terms of information content:  $X$  is 1-random if it is *incompressible*, that is, the initial segments  $(X(0), X(1), \dots, X(n))$  have high Kolmogorov [20] or Levin-Chaitin [11, 23] complexity. A third definition may be given in terms of martingales:  $X$  is 1-random if it is *unpredictable*, that is, there is no effective

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martingale for which one can obtain unbounded capital by betting on the values of  $X$  [33]. These three versions have been shown by Schnorr [31] to be equivalent. This demonstrates the robustness of the concept of Martin-Löf randomness. Other interesting notions of algorithmic randomness have been studied, although in some cases, formulations are only given for one or perhaps two versions. For a thorough study of the area of algorithmic randomness, the reader is directed to three excellent recently published books: Downey and Hirschfeldt [15], Nies [29] and Li and Vitanyi [24].

This paper presents a notion of bounded randomness for some classes of sub-computable functions which do not have access to a universal machine for that class. We begin in Section 2 by stating our definitions for the class of primitive recursive functions and define a new notion of *bounded primitive recursive randomness* (BP randomness). We show that there are three equivalent definitions of BP randomness: one in terms of measure, one in terms of compressibility, and one in terms of martingales. For measure, a *bounded primitive recursive test* will be a primitive recursive sequence of clopen sets  $(U_n)_{n \geq 0}$  such that  $U_n$  has measure  $\leq 2^{-n}$ . We define  $X$  to be BP *random* if it does not belong to  $\bigcap_{n \geq 0} U_n$  for any such test. For compressibility, we say that  $X$  is BP *compressed* by a primitive recursive machine  $M$  if there is a primitive recursive function  $f$  such that  $C_M(X \upharpoonright f(c)) \leq f(c) - c$  for all  $c$ , where  $C_M$  is a primitive recursive analogue of Kolmogorov complexity. We show that  $X$  is BP random if and only if  $X$  is not BP compressed by any primitive recursive machine. We will also consider process machines and the resulting notion of process complexity as studied recently by Day [14].

For martingales, we say that a primitive recursive martingale  $d$  *succeeds* on a sequence  $X$  if there is a primitive recursive function  $f$  such that  $d(X \upharpoonright f(n)) \geq 2^n$  for each  $n$ . Thus  $d$  makes us rich betting on  $X$ , and  $f$  tells us how fast this happens. We show that  $X$  is BP random if and only if there is no primitive recursive martingale which succeeds on  $X$ .

As we shall see, the presence of the auxiliary functions  $f$  in the definition of  $X$  being BP-compressed by a primitive recursive machine and in the definition of when a primitive recursive martingale succeeds on a sequence  $X$  is key to our ability to prove that the three definitions of BP randomness are all equivalent. These definitions can be adapted to define a notion of bounded randomness for other classes of sub-computable functions and if such a class satisfies certain closure conditions, then our proofs of the equivalence of the three notions of primitive recursive randomness can be adapted to prove similar results for that class of sub-computable functions. However some care has to be taken in the definitions of the analogue of the auxiliary functions  $f$  described above. As an example, in the last part of the paper, we define a notion of bounded polynomial space randomness, called BPS randomness. We show that the results obtained for BP randomness carry over, after suitable modifications, to BPS randomness.

One motivation for the definition of BPS randomness is that it gives a natural notion of randomness for polynomial space which has equivalent formulations in terms of compressibility, in terms of statistical tests, and in terms of martingales. Moreover, our approach suggests a general alternative approach to defining randomness via martingales for various classes of sub-computable functions by adding the requirement that the rate of success of a strategy is measured by a function in that class. As we discuss below, our notion of bounded randomness differs from the well-studied notions of resource-bounded randomness which have been developed over the past twenty-five years. Finally, our general approach to defining randomness on sub-computable classes of functions should be useful for the study of resource-bounded trees and effectively closed sets.

The terms *bounded randomness* and *finite randomness* are sometimes used to refer to versions of randomness given by tests in which the c.e. open sets are in fact clopen. The term “finite” comes from the fact that any clopen set  $U$  is the finite union of intervals  $U = [\sigma_1] \cup \dots \cup [\sigma_k]$ . Our notion of BP randomness is “bounded” in this sense. Kurtz randomness [21], also referred to as weak randomness, falls into this category. A real  $X$  is Kurtz random if it does not belong to any  $\Pi_1^0$  class  $Q$  of measure zero. But any  $\Pi_1^0$  class may be effectively expressed as a decreasing intersection of clopen sets  $Q = \bigcap_n Q_n$  where the clopen sets  $Q_n$  are unions of intervals of length  $n$ . If  $\mu(Q) = 0$ , it is easy to find a subsequence  $U_i = Q_{n_i}$  with  $\mu(U_i) \leq 2^{-i}$  and thus  $(U_n)_{n \geq 0}$  is a bounded Martin-Löf test. From this it follows immediately that if our definition of BP random is modified to use computable functions instead of primitive recursive functions, then it becomes equivalent to Kurtz randomness. Kurtz randomness has been well-studied, and is already known to have equivalent definitions in terms of Martin-Löf randomness, Kolmogorov compressibility, and effective martingales; for this see Wang [34], Downey, Griffiths and Reid [16], and Bienvenu and Merkle [6]. Uniform relative Kurtz randomness is studied by Kihara and Miyabe [18], who proved van Lambalgen’s theorem holds for their definition. Another special type of bounded randomness was recently studied by Brodhead, Downey and Ng [8].

As shown by Wang [34], Kurtz random reals may not be stochastic in the sense of Church [13]. For example, it need not be the case that the number of occurrences of 0’s in a Kurtz random sequence  $X$  tends to  $1/2$  in the limit. This can also happen for our BP random reals. Indeed, we construct a recursive real which is BP random but not stochastic. However, in Section 2.4, we show that BP random sets do satisfy a weak “infinitely often” version of the stochastic property.

A lot of work has been done on various notions of resource-bounded randomness. One of the first approaches to resource-bounded randomness was via the above-mentioned stochastic property of typical reals. It is expected that for a random real, the relative densities of 0’s and 1’s should be equal in the limit. We identify a set  $A$  of natural numbers with its characteristic function and in those terms we expect that  $\lim_n (\text{card}(A \cap [[n]])/n) = \frac{1}{2}$  where  $[[n]] = \{0, 1, \dots, n-1\}$ . Levin [23] defined a notion of primitive randomness for a set  $A$  to mean that for every primitive recursive set  $B$ , the set  $A \cap B$  is stochastic relative to  $B$ , and constructed a recursive set that is primitive random. Di Paola [30] studied similar notions of randomness in the Kalmar hierarchy of elementary functions. Wilber [35] defined a set  $A$  to be P random if, for every PTIME set  $B$ , the sets  $A$  and  $B$  agree on a set of density  $\frac{1}{2}$ , and constructed an exponential time computable P random set.

The literature of computational complexity contains many papers on random number generators and cryptography which examine various notions of pseudorandomness. For example, Blum and Micali [7] gave a weak definition of pseudorandom sequences in which a randomly generated sequence is said to be pseudorandom if it meets all PTIME statistical tests. Ko [19] gave definitions of randomness with respect to polynomial time and polynomial space complexity which are in the tradition of algorithmic randomness as established by Levin, Martin-Löf, Chaitin, and Schnorr. Ko’s notion of polynomial space randomness has equivalent formulations in terms of tests and in terms of compressibility (namely, the equivalent classes PSR1 and PSR2 in [19]). Ko also shows PSR1-/PSR2-random reals satisfy strong stochastic properties. The intuition behind Ko’s polynomial space randomness is very different from the motivation of our BPS randomness. Indeed, Ko defines a notion of randomness based on the intuition that no polynomial space test can “reject the hypothesis that  $x$  is random on the significance level of  $2^{-m}$ ” [19]. However, this is exactly the notion

of polynomial space test that we *are* adopting. For instance, Ko's definition of PSR2 is based on compressibility using a universal polynomial space machine  $M$  and states that a real  $X$  is (PSPACE) compressed if, for every  $k$ , there exists infinitely many  $n$  such that  $K_M(X \upharpoonright n) < n - (\log n)^k$ , where  $K$  is a notion of Kolmogorov complexity. (We have not stated the definition fully: see [19] for the correct, precise definition.) In contrast, our definition of BPS states, loosely speaking, that  $X$  is (PSPACE) compressed provided there is some  $k$  such that  $K_M(X \upharpoonright n^k) \leq n^k - n$  for all sufficiently large  $n$ . (See Section 3 for the actual definition; in particular, unlike Ko, we do not use a universal machine  $M$ ). Clearly every real which is polynomial space random in Ko's sense is also BPS random. The converse does not hold, as BPS randomness does not have as strong stochastic properties. For this, see the discussion after Theorem 3.21.

Lutz [26], building on Schnorr [31], defined an important notion of resource-bounded randomness in terms of martingales. For the special case of polynomial space, Lutz defined that a real  $X$  is PSPACE random if there is no PSPACE martingale which succeeds on  $X$ . This differs from our definition of BPS randomness in that there is no auxiliary function  $f$  which bounds the time-to-success for the martingale. We shall see that this is a strictly stronger notion than BPS random. Clearly, every PSPACE random real is automatically BPS random. On the other hand, there are BPS random reals which are not PSPACE random in the sense of Lutz. This will be an immediate consequence of our Theorem 3.21, which states that BPS random reals do not enjoy as strong stochastic properties as are known to hold for PSPACE random reals. Lutz further defined that a set  $\mathcal{X}$  of reals has PSPACE measure one if there is no PSPACE martingale which succeeds on every element of  $\mathcal{X}$ . Then almost every EXPSPACE real is random, and this can be used to study properties of EXPSPACE reals by examining whether the set of EXPSPACE reals with the property has measure one. Buhrman and Longpre [9] gave a rather complicated equivalent formulation of PSPACE randomness in terms of compressibility. Lutz's notion of complexity theoretic randomness concept has been developed further by [1, 2, 3, 28]. Shen et al. [12] have recently studied on-line complexity and randomness.

Ville's theorem is a fundamental property of Martin-Löf random reals and states that any effective subsequence of a random sequence is also random. We prove an analogue of Ville's theorem for BP randomness.

Another fundamental property for random reals is van Lambalgen's theorem, which states that the join  $A \oplus B$  of two random sets is random if and only if  $B$  is random and  $A$  is random relative to  $B$ . We define a natural notion of *relative BP randomness* which still has three equivalent formulations. We prove one direction of the analogue of van Lambalgen's theorem, showing that if  $A$  is BP random relative to  $B$  and  $B$  is BP random relative to  $A$ , then  $A \oplus B$  is BP random. However, we give a counterexample for the converse, by showing there exist  $A$  and  $B$  such that  $A$  is BP random relative to  $B$  and  $B$  is BP random relative to  $A$ , but  $A \oplus B$  is not BP random. This corrects Theorem 6 of [10] which claimed that both directions of van Lambalgen's theorem hold for BP and BPS randomness.

For the case of bounded BPS randomness, we again present equivalent notions, one in terms of compressibility, one in terms of measure, and one in terms of martingales. In this case, polynomial growth rate functions are used to bound the rate of compressibility, the size of the measure-based tests, and the success rate of the martingales. The compressibility definition is shown to be equivalent whether defined in terms of prefix-free or non-prefix-free functions. We prove that one direction of van Lambalgen's theorem holds for relative

randomness of bounded BPS randomness, whereas the other direction fails. We also discuss stochastic properties of bounded BPS random sequences.

The outline of this paper is as follows. Section 2 studies BP randomness and shows the equivalence of our three versions. We construct a computable real which is BP random. We prove an analogue of Ville’s theorem for primitive recursive subsequences of BP random reals. We also define a notion of relative randomness and discuss the analogue of van Lambalgen’s theorem. Section 3 presents three equivalent notions of bounded polynomial space randomness (which we call “BPS randomness” to distinguish it from earlier different definitions); and extends all the constructions of Section 2 to bounded BPS randomness. Section 4 states our conclusions and some directions for further research.

A preliminary version of this paper was published in [10]. The present paper has the following improvements. It includes all of the proofs for the case of BPS randomness which were omitted in the conference version. It presents a new definition of BP randomness and BPS randomness using process machines and the proof of their equivalence with the other definitions. It proves the equivalence of BPS randomness and weak BPS randomness. It gives a proof of the intuitive fact that prefix-freeness is not needed for the Kolmogorov complexity version of BP and BPS randomness. It gives a proof that only one direction of van Lambalgen’s theorem holds for BP and BPS randomness. This corrects Theorem 6 of [10], which claimed that both directions of van Lambalgen’s Theorem held for BP and BPS. It contains an improvement in the construction of a BP random real which significantly lowers the complexity from  $2^{2^n}$  to  $n^{\epsilon \log n}$ .

## 2. BOUNDED PRIMITIVE RECURSIVE RANDOMNESS

This section defines the three notions of primitive recursive randomness, Kolmogorov BP randomness, Martin-Löf BP randomness, and martingale BP randomness, and proves their equivalence. Hence we say that a real  $X$  is BP random if it satisfies these three definitions. Subsection 2.3 introduces a notion of prefix-free BP randomness, and shows it is also equivalent to BP randomness. We also prove a simple analogue of Ville’s theorem and discuss statistical tests and BP randomness.

### 2.1. Three definitions for BP randomness.

2.1.1. *Martin-Löf BP randomness.* We work with the usual alphabet  $\Sigma = \{0, 1\}$  and the corresponding set  $\{0, 1\}^*$  of finite strings and the Cantor space  $\{0, 1\}^\omega$  of infinite sequences, but our results hold for any finite alphabet. In this section, we study primitive recursive functions  $M : \Sigma^* \rightarrow \Sigma^*$ . We code finite strings as numbers in order to define these primitive recursive functions using primitive recursive coding and decoding functions. The code  $c(\sigma)$  of a finite sequence  $\sigma = \sigma_1 \cdots \sigma_n \in \{0, 1\}^*$  is just the natural number whose binary expansion is  $1\sigma_1 \cdots \sigma_n$ . Given a nonempty finite set  $S = \{\sigma^{(1)}, \dots, \sigma^{(k)}\}$  of strings in  $\{0, 1\}^*$  such that  $c(\sigma^{(1)}) < \dots < c(\sigma^{(k)})$ , the code  $C(S)$  of  $S$  is defined to be the natural number  $n$  whose ternary expansion is  $2c(\sigma^{(1)})2 \cdots 2c(\sigma^{(k)})$ . We let  $\emptyset$  denote the empty string. The set of infinite 0/1 sequences is denoted  $\{0, 1\}^\omega$ ; these are also called *reals*. An  $X \in \{0, 1\}^\omega$  is also viewed as a set, namely the set  $\{i : X(i) = 1\}$ .

For any string  $\sigma \in \{0, 1\}^*$ ,  $|\sigma|$  denotes the length of  $\sigma$ . For  $X \in \{0, 1\}^* \cup \{0, 1\}^\omega$ , we write  $\sigma \sqsubset X$  if  $\sigma$  is an initial segment of  $X$ , and let  $[\sigma] = \{X \in \{0, 1\}^\omega \text{ such that } \sigma \sqsubset X\}$ . For a set  $G$  of strings in  $\{0, 1\}^*$ , we let  $[G] = \bigcup\{[\sigma] : \sigma \in G\}$ . We say a sequence  $(U_n)_{n \in \mathbb{N}}$  of

clopen sets is a *primitive recursive sequence* if there is a primitive recursive function  $f$  such that for all  $n$ ,  $f(n)$  is a code of a finite set  $G_n = \{\sigma_{1,n}, \dots, \sigma_{k(n),n}\}$  such that  $U_n = [G_n]$ .

We define a *primitive recursive test* to be a primitive recursive sequence  $(U_n)_{n \geq 0}$  of clopen sets such that, for each  $n$ ,  $\mu(U_n) \leq 2^{-n}$ . Without loss of generality, we may assume that there is a primitive recursive function  $g$  such that  $g(n)$  is a code of a finite set  $G_n = \{\tau_{1,n}, \dots, \tau_{r(n),n}\}$  such that  $U_n = [G_n]$  and  $|\tau_{1,n}| = \dots = |\tau_{r(n),n}| = \ell(n)$  where  $r$  and  $\ell$  are also primitive recursive functions. It follows that there is a primitive recursive function  $m$  such that  $m(n)$  codes the measure  $\mu(U_n)$  as a dyadic rational. Since the measures  $\mu(U_n)$  may be computed primitive recursively, one could equivalently define a primitive recursive test to be a primitive recursive sequence  $(V_n)_{n \geq 0}$  such that  $\lim_n \mu(V_n) = 0$  and there is a primitive recursive function  $f$  such that, for each  $p$ ,  $\mu(V_{f(p)}) \leq 2^{-p}$ .

Observe that  $\bigcap_n U_n$  is a  $\Pi_1^0$  class of measure zero, so any primitive recursive test is a Kurtz test and hence also a Schnorr test.

**Definition 2.1.** An infinite sequence  $X \in \{0,1\}^\omega$  is *Martin-Löf BP random* if  $X$  passes every primitive recursive test, that is, for every primitive recursive test  $(U_n)_{n \geq 0}$ , there is some  $n$  such that  $X \notin U_n$ .

By the remarks above, every Kurtz random real is Martin-Löf BP random.

**Proposition 2.2.**  $X$  is Martin-Löf BP random if and only if there is no primitive recursive sequence  $(U_n)_{n \geq 0}$  of clopen sets with  $\mu(U_n) = 2^{-n}$  such that  $X \in \bigcap_n U_n$ .

*Proof.* The if direction is immediate. Now suppose that there is a primitive recursive sequence  $(V_n)_{n \geq 0}$  such that  $\mu(V_n) \leq 2^{-n}$  and  $X \in \bigcap_n V_n$ . Let  $V_n = \bigcup_{\sigma \in G_n} [\sigma]$  where  $G_n \subseteq \{0,1\}^{\ell(n)}$  for some primitive recursive function  $\ell(n)$ , where  $\ell(n) \geq n$  for all  $n$ . Then  $\mu(V_n) = \text{card}(G_n)/2^{\ell(n)} \leq 2^{-n}$ . Define  $H_n$  to be  $G_n$  together with  $2^{\ell(n)-n} - \text{card}(G_n)$  additional strings of length  $\ell(n)$  and let  $U_n = [H_n]$ . Then for each  $n$ ,  $X \in U_n$  and  $\mu(U_n) = 2^{-n}$ .  $\square$

We also need the notion of a *weak primitive recursive test*. A *weak primitive recursive test*  $(U_n)_{n \geq 0}$  is a primitive recursive test for which there are a primitive recursive sequence  $(G_n)_{n \geq 0}$  and a primitive recursive function  $\ell$  such that for each  $n$ ,  $U_n = [G_n]$  and, for all  $\tau \in G_n$ ,  $|\tau| = \ell(n)$  and  $\mu(U_{n+1} \cap [\tau]) \leq \frac{1}{2}\mu([\tau])$ .

By definition, every weak primitive recursive test  $(U_n)_{n \geq 0}$  is also a primitive recursive test. Conversely, we can convert a primitive recursive test  $(U_n)_{n \geq 0}$  into a weak primitive recursive test as follows. First, we may assume that  $U_{n+1} \subseteq U_n$  for each  $n$ , since the sequence  $W_n = \bigcap_{i \leq n} U_i$  is also a primitive recursive test with  $\mu(W_n) \leq \mu(U_n) \leq 2^{-n}$ . Next suppose  $U_n = [\tau_{1,n}] \cup \dots \cup [\tau_{k(n),n}]$  where there is a primitive recursive function  $\ell$  such that  $|\tau_{i,n}| = \ell(n)$  for all  $i$ , so each interval  $[\tau_{i,n}]$  has measure exactly  $2^{-\ell(n)}$ . The clopen set  $U_{\ell(n)+1}$  has a total measure  $\leq 2^{-\ell(n)-1}$ , so  $\mu(U_{\ell(n)+1} \cap [\tau_{i,n}]) \leq \mu(U_{\ell(n)+1}) \leq \frac{1}{2}\mu([\tau_{i,n}])$ . Then we can define a primitive recursive weak test  $(V_n)_{n \geq 0}$  as follows. Let  $h(0) = 0$  and let  $V_0 = U_0$ . Then let  $h(1) = \ell(0) + 1$  and  $V_1 = U_{h(1)}$ . In general for  $n > 1$ , we let  $h(n+1) = \ell(h(n)) + 1$  and let  $V_{n+1} = U_{h(n+1)}$ . The sequence  $V_0, V_1, \dots$  will be a weak primitive recursive test. Since the sequence  $(V_n)_{n \geq 0}$  is a subsequence of the original sequence  $(U_m)_{m \geq 0}$ , it follows that  $\bigcap_n V_n = \bigcap_n U_n$ , so that  $X$  passes the weak test  $(V_n)_{n \geq 0}$  if and only if it passes the original test.

We have established the following.

**Proposition 2.3.**  $X$  is Martin-Löf BP random if and only if it passes every weak primitive recursive test.

2.1.2. *Kolmogorov BP randomness.* Let  $M : \{0,1\}^* \rightarrow \{0,1\}^*$  be a primitive recursive function. Let  $C_M(\tau)$  be the length  $|\sigma|$  of the shortest string  $\sigma$  such that  $M(\sigma) = \tau$ , that is, the length of the shortest  $M$ -description of  $\tau$ . Notice that we are using plain and not prefix-free complexity. We say that  $X$  is *primitive recursively compressed* by  $M$  if there exists a primitive recursive function  $f$  such that, for every  $c \in \mathbb{N}$ ,  $C_M(X \upharpoonright f(c)) \leq f(c) - c$ .

**Definition 2.4.** An infinite sequence  $X \in \{0,1\}^\omega$  is *Kolmogorov BP random* if it is not primitive recursively compressed by any primitive recursive function  $M : \{0,1\}^* \rightarrow \{0,1\}^*$ .

Our definition of primitive recursive compressibility is a natural effective analogue of the usual definition of Kolmogorov compressibility which says that, for every  $c \in \mathbb{N}$ , there exists  $n$  such that  $C_M(X \upharpoonright n) \leq n - c$ . That is, given the formulation “for every  $c$ , there exists  $n$ ”, there must exist a function  $f$  which computes a value  $n = f(c)$  for each input  $c$ . The complexity of the function  $f$  calibrates the difficulty of computing a value of  $n$  for an arbitrary input  $c$ . In the setting of primitive recursive computability, it is reasonable to require that  $f$  be primitive recursive. In addition, this makes the definition equivalent to the definition given above for BP Martin-Löf randomness.

The function  $f$  can be assumed to be strictly increasing without loss of generality; for this see the remark after the proof of Theorem 2.7.

Of course, one defines Kolmogorov randomness in terms of prefix-free complexity  $K_M$  since there are no infinite Kolmogorov random sequences for plain complexity. The definition of Kolmogorov BP random uses plain complexity since every primitive recursive function is total so there are no prefix-free machines. However, Section 2.3 considers a version of prefix-free complexity for primitive recursive functions which uses primitive recursive functions  $M$  such that  $M(\sigma)$  may diverge. This will be done by introducing a new symbol  $\infty$  as a possible output of  $M(\sigma)$  to signify that  $M(\sigma)$  diverges. It will not be hard to show that this makes no difference.

We also consider process machines and the resulting notion of process complexity. A partial computable function  $M : \{0,1\}^* \rightarrow \{0,1\}^*$  is said to be a *process machine* if, whenever  $\tau \sqsubset \tau'$  and  $\tau, \tau' \in \text{Dom}(M)$ , then  $M(\tau) \sqsubset M(\tau')$ , so that  $M$  is extension preserving when defined. A process machine  $M$  is a *strict process machine* if, whenever  $\tau \sqsubset \tau' \in \text{Dom}(M)$ , then  $\tau \in \text{Dom}(M)$ , so the domain of  $M$  is closed under prefixes. A strict process machine  $M$  is a *quick process machine* if  $M$  is total and there is an order function  $h$  such that, for all  $\tau \in \{0,1\}^*$ ,  $|M(\tau)| \geq h(|\tau|)$ . Recall that  $h : \mathbb{N} \rightarrow \mathbb{N}$  is an *order function* provided  $h$  is non-decreasing and  $\lim_n h(n) = \infty$ . A strict process machine  $M$  is called a *quick process BP machine* if the order function  $h$  is primitive recursive. Finally,  $X \in \{0,1\}^\omega$  is *quick process BP random* provided  $X$  is not primitively recursively compressed by any quick process BP machine.

The definition of a process machine is due to Levin and Zonkin [22] and a similar notion was defined by Schnorr [32]. Day [14] gives characterizations of computable randomness, Schnorr randomness, and weak randomness using quick process machines. Theorem 2.7 below shows that  $X$  is Kolmogorov BP random if and only if  $X$  is quick process BP random.

**2.1.3. Martingale BP randomness.** A martingale is a function  $d : \{0, 1\}^* \rightarrow \mathbb{Q} \cap [0, \infty)$  such that  $d(\emptyset) = 1$  and, for all  $\sigma \in \{0, 1\}^*$ ,  $d(\sigma) = (d(\sigma \frown 0) + d(\sigma \frown 1))/2$ . Of course, any primitive recursive martingale is also a computable martingale. We say that the martingale  $d$  *succeeds primitive recursively on  $X$*  if there is a primitive recursive function  $f$  such that, for all  $n$ ,  $d(X \upharpoonright f(n)) \geq 2^n$ . (Of course, we could replace  $2^n$  here with any strictly increasing primitive recursive function.) In general, a martingale  $d$  is said to succeed on  $X$  if  $\limsup_n d(X \upharpoonright n) = \infty$ , that is, for every  $n$ , there exists  $m$  such that  $d(X \upharpoonright m) \geq 2^n$ . Thus our definition is an effectivization of the usual definition with a primitive recursive function  $f$  which witnesses some point where  $d$  will return  $2^n$ . A martingale could be thought of as a financial advisor who guarantees the client's eventual wealth; effective success of the martingale means that the advisor can predict when the client will reach a given level of wealth. Again, the requirement that  $f$  be primitive recursive is necessary to have the equivalence with BP Martin-Löf randomness.

**Definition 2.5.**  $X$  is *martingale BP random* if there is no primitive recursive martingale which succeeds primitive recursively on  $X$ .

If  $X$  is *not* martingale BP random, then there is a computable martingale which succeeds primitive recursively on  $X$  and thus certainly succeeds on  $X$ , so  $X$  is not computably random. Hence every computably random real is also a martingale BP random real.

The definition of martingale BP random real has the following equivalent formulations.

**Proposition 2.6.** The following are equivalent.

- (1):  $X$  is martingale BP random.
- (2): There do not exist a primitive recursive martingale  $d$  and a primitive recursive function  $f$  such that, for every  $n$ , there exists  $m \leq f(n)$  such that  $d(X \upharpoonright m) \geq 2^n$ .
- (3): There do not exist a primitive recursive martingale  $d$  and a primitive recursive function  $f$  such that  $d(X \upharpoonright m) \geq 2^n$  for all  $n$  and all  $m \geq f(n)$ .

*Proof.* The implications (1)  $\Rightarrow$  (3) and (2)  $\Rightarrow$  (1) are obvious. The proof of (3)  $\Rightarrow$  (2) uses the idea of a *savings account* as formulated in [15, 29]. Let the martingale  $d$  and function  $f$  be as in (2). We shall modify these to form a new pair  $d'$  and  $f'$  which satisfy (3). The intuition is that  $d(\tau)$  is modified so that whenever  $d(\tau) \geq 2^{n+1}$ , then one half of the working capital, as represented by  $d(\tau)$ , is transferred to a savings account. That is, loosely speaking, every time the capital is doubled, one half of it is placed into the savings account. Formally, we define  $d'(\tau)$  as follows. Given  $\tau \in \{0, 1\}^*$ , define  $\ell_n^\tau$  to equal the least value, if any, such that  $d(\tau \upharpoonright \ell_n^\tau) \geq 2^{n+1}$ . Let  $\tau_n$  equal  $\tau \upharpoonright \ell_n^\tau$ , and let  $n(\tau)$  be the least value of  $n$  for which  $\ell_n^\tau$  is undefined. Thus,  $\tau_{n(\tau)-1} \sqsubset \tau$ , and  $\tau_{n(\tau)}$  is undefined. Note  $n(\emptyset) = 0$ . The martingale  $d'(\tau)$  is defined by

$$d'(\tau) = \frac{d(\tau)}{2^{n(\tau)}} + \sum_{i=0}^{n(\tau)-1} \frac{d(\tau_i)}{2^{i+1}}.$$

As the reader can easily verify,  $d'$  is a martingale. The intuition is that the summation on the righthand side of the equation equals the capital placed in the savings account, and  $d(\tau)/2^{n(\tau)}$  is the remaining working capital.

Since  $d(\tau_i) \geq 2^{i+1}$ , we have  $d(\tau) \geq n$  whenever  $\ell_n^\tau$  is defined; in particular, this holds whenever  $d(\tau') \geq 2^{n+1}$  for some  $\tau' \sqsubset \tau$ . Finally, define  $f'(n)$  to equal  $f(2^{n+1})$ . Let  $X \in \{0, 1\}^\omega$  and  $m \geq f'(n)$ . Since  $m \geq f(2^{n+1})$ ,  $d(X \upharpoonright m) \geq 2^{2^{n+1}}$ . Therefore,  $d'(X \upharpoonright m) \geq 2^n$ . This establishes that the property of (3) holds for  $d'$  and  $f'$ .  $\square$



**2.2. Equivalences.** The main result of this section is the next two theorems showing that the three versions of BP random described above are equivalent.

**Theorem 2.7.** The following statements are equivalent for  $X \in \{0, 1\}^\omega$ :

- (1):  $X$  is Martin-Löf BP random.
- (2):  $X$  is Kolmogorov BP random.
- (3):  $X$  is quick process BP random.

*Proof.* The implication (2)  $\Rightarrow$  (3) is immediate.

(1)  $\Rightarrow$  (2): Suppose (2) is false and  $X$  is not Kolmogorov BP random. Then there exist primitive recursive  $M$  and  $f$  such that  $C_M(X \upharpoonright f(c)) \leq f(c) - c$  for all  $c \in \mathbb{N}$ .

Let  $U_c = \{Y \in \{0, 1\}^\omega : C_M(Y \upharpoonright f(c+1)) \leq f(c+1) - c - 1\}$ . We claim this is a uniformly primitive recursive sequence of clopen sets. To see this, let

$$G_c = \{M(\sigma) : \sigma \in \{0, 1\}^{\leq f(c+1)-c-1}\} \cap \{0, 1\}^{f(c+1)}.$$

Then  $U_c = [G_c]$ , and this expresses  $(U_c)_{c \geq 0}$  as a primitive recursive sequence of clopen sets.

We also claim that  $\mu(U_c) \leq 2^{-c}$ . For this, fix  $c$  and let  $U_c = [\tau_1] \cup [\tau_2] \cup \dots \cup [\tau_k]$ , for distinct  $\tau_i \in \{0, 1\}^{f(c+1)}$ . Thus there exist  $\sigma_1, \dots, \sigma_k$  such that  $|\sigma_i| \leq f(c+1) - c - 1$  and  $M(\sigma_i) = \tau_i$  for each  $i \leq k$ . Since there are only  $2^{f(c+1)-c-1}$  strings of length  $\leq f(c+1) - c - 1$ , we have  $k < 2^{f(c+1)-c}$ . Since for each  $i$ ,  $\mu([\tau_i]) = 2^{-f(c+1)}$ ,

$$\mu(U_c) = k \cdot 2^{-f(c+1)} < 2^{f(c+1)-c} \cdot 2^{-f(c+1)} = 2^{-c}.$$

Therefore,  $(U_c)_c$  is a primitive recursive test. By assumption,  $X \in U_c$  for all  $c \geq 0$ , so  $X$  is not Martin-Löf BP random.

(3)  $\Rightarrow$  (1): Suppose that  $X$  is not Martin-Löf BP random. Then there exist primitive recursive functions  $g$ ,  $k$ , and  $f$  so that for all  $c \geq 0$ ,  $g(c)$  is a code of a set  $G_c \subseteq \{0, 1\}^{f(c)}$  with cardinality  $k(c)$ , such that if  $U_c = [G_c]$  then  $\mu(U_c) \leq 2^{-c}$  and such that  $X \in \bigcap_c U_c$ . Furthermore, we may assume by Proposition 2.3 that this is a weak test, so that, for each  $\sigma \in G_c$ ,  $\mu([\sigma] \cap U_{c+1}) \leq \frac{1}{2}\mu([\sigma])$ . By the proof of Proposition 2.3, we may assume that  $U_{c+1} \subset U_c$ . We may assume without loss of generality that for each  $c$ ,  $f(c+1) - (c+1) > f(c) - c$ . This is because we may always break each  $[\tau]$  into  $[\tau \frown 0] \cup [\tau \frown 1]$  to increase  $f(c)$  by one, if necessary. Also w.l.o.g.,  $f(0) = 0$ , so  $U_0 = \{0, 1\}^\omega$ .

We define a quick process BP machine  $M$  such that  $C_M(\sigma) \leq f(c) - c$  for all  $c$  and all  $\sigma \in G_c$ . Since  $X \upharpoonright f(c) \in G_c$ , we have  $C_M(X \upharpoonright f(c)) \leq f(c) - c$  for all  $c$ . The machine  $M$  is defined in stages as follows.

At stage  $c = 1$ , we have  $\mu(U_1) \leq \frac{1}{2}$ , and since  $\mu(U_1) = k(1) \cdot 2^{-f(1)}$ , it follows that  $k(1) \leq 2^{f(1)-1}$ . Let  $G_1 = \{\tau_1, \dots, \tau_{k(1)}\}$ . Take the lexicographically first  $k$  strings  $\sigma_1, \dots, \sigma_{k(1)}$  of length  $f(1) - 1$ , and define  $M(\sigma_i) = \tau_i$ . To make  $M$  a total function, the remaining strings of length  $f(1) - 1$  are all mapped to  $0^{f(1)}$ , and all strings of length  $< f(1) - 1$  are mapped to the empty string  $\emptyset$ .

Observe that for all strings  $\sigma$  of length  $< f(1) - 1$ ,  $|M(\sigma)| = 0$  and for all strings  $\sigma$  of length  $f(1) - 1$ ,  $|M(\sigma)| = f(1)$ .

After stage  $c$ , we have defined  $M(\sigma)$  for all strings  $\sigma$  of length  $\leq f(c) - c$  so that  $M$  is extension-preserving and such that, for each  $b \leq c$  and each  $\tau \in G_b$ , there exists a  $\sigma$  of length  $f(b) - b$  with  $M(\sigma) = \tau$ . Furthermore, for any  $\sigma$ , if  $f(b-1) - b + 1 \leq |\sigma| < f(b) - b$ , then  $|M(\sigma)| = f(b-1)$ , and if  $|\sigma| = f(c) - c$ , then  $|M(\sigma)| = f(c)$ .

At stage  $c+1$ , we define  $M(\sigma)$  when  $f(c) - c < |\sigma| \leq f(c+1) - c - 1$ . For each  $\nu$  of length  $f(c) - c$  we work on the extensions of  $\nu$  independently. We set  $M(\sigma) = M(\nu)$

for any extension  $\sigma$  of  $\nu$  of length  $< f(c+1) - c - 1$ . If  $M(\nu) \notin G_c$ , then we let  $M(\sigma) = M(\nu) \frown 0^{f(c+1)-1-f(c)}$  for all  $\sigma \sqsupset \nu$  with  $|\sigma| = f(c+1) - c - 1$ . Otherwise,  $M(\nu) \in G_c$  and let  $H_\nu$  be the members of  $G_{c+1}$  which extend  $M(\nu)$ :

$$H_\nu = \{\rho \in \{0, 1\}^{f(c+1)-f(c)} : M(\nu) \frown \rho \in G_{c+1}\}.$$

Since  $(U_c)_c$  is a weak test,  $\mu(H_\nu) \leq \frac{1}{2}$  and so  $k_\nu = |H_\nu| \leq 2^{f(c+1)-f(c)-1}$ . Enumerate  $H_\nu$  in lexicographic order as  $\{\rho_1, \dots, \rho_{k_\nu}\}$ . Let  $\sigma_i$  be the  $i$ -th string of length  $f(c+1) - f(c) - 1$  in lexicographical order with  $i \leq k_\nu$  and set  $M(\nu \frown \sigma_i) = M(\nu) \frown \rho_i$ . For all other  $\sigma \in \{0, 1\}^{f(c+1)-f(c)-1}$ , set  $M(\nu \frown \sigma) = M(\nu) \frown 0^{f(c+1)-f(c)}$ .

It is clear that  $M$  continues to be extension-preserving. Since  $U_{c+1} \subset U_c$ , each  $\tau \in G_{c+1}$  is equal to  $M(\nu) \frown \rho$  for some  $\nu$  and some  $\rho \in H_\nu$ . It follows that  $\tau = M(\sigma)$  for some  $\sigma$  of length  $f(c+1) - c - 1$ . Finally, for any  $\sigma$ , if  $f(c) - c \leq |\sigma| < f(c+1) - c - 1$ , then  $|M(\sigma)| = f(c)$  and also if  $|\sigma| = f(c+1) - c - 1$ , then  $|M(\sigma)| = f(c+1)$ .

To see that  $M$  is primitive recursive, observe that, since  $f(c+1) - c - 1 > f(c) - c$  for all  $c$ , we have  $f(c) - c \geq c$ . Thus, given a string  $\sigma$  of length  $m$ , we need check only values  $c \leq m$  to find the least  $c$  such that  $m \leq f(c) - c$ . Then we simply run the process above for  $c$  stages to compute  $M(\sigma)$ .

To verify that  $M$  is a quick process machine, we define a function  $h(m)$  so that  $|M(\tau)| \geq h(|\tau|)$  for all  $\tau$ : First let  $h'(m)$  be the least  $c \leq m$  such that  $m < f(c+1) - c - 1$  and then let  $h(m) = f(h'(m))$ .

By assumption,  $X \in U_c$  for every  $c$ , so  $X \upharpoonright f(c) = \tau$  for some  $\tau \in G_c$  hence  $M(\sigma) = \tau$  where  $|\sigma| = f(c) - c$ . It follows that  $C_M(X \upharpoonright f(c)) = f(c) - c$ . Hence,  $X$  is not quick process BP random.  $\square$

The proof of (3)  $\Rightarrow$  (1) above showed that the function  $f$  giving the length of strings for the Martin-Löf tests can be strictly increasing without loss of generality. The same function  $f$  was then used for showing that  $X$  can be primitive recursively compressed. It follows that in the definition of Kolmogorov BP random,  $f$  may be assumed to be strictly increasing without loss of generality.

**Theorem 2.8.** The following statements are equivalent for  $X \in \{0, 1\}^\omega$ .

- (1):  $X$  is Martin-Löf BP random.
- (2):  $X$  is martingale BP random.

*Proof.* (1)  $\Rightarrow$  (2): Suppose that  $X$  is not martingale BP random. Then there is a primitive recursive martingale  $d$  and a primitive recursive function  $f$  such that, for all  $n$ ,  $d(X \upharpoonright f(n)) \geq 2^n$ . Let  $G_n = \{\tau \in \{0, 1\}^{f(n)} : d(\tau) \geq 2^n\}$  and  $U_n = [G_n]$ . The sequence  $(U_n)_{n \geq 0}$  is a primitive recursive sequence of clopen sets, and  $X \in \bigcap_n U_n$ .

Since  $d$  is a martingale and  $d(\emptyset) = 1$ ,  $\sum_{|\tau|=m} d(\tau) \leq 2^m$ . It follows that there are at most  $2^{f(n)-n}$  strings  $\tau \in \{0, 1\}^{f(n)}$  such that  $d(\tau) \geq 2^n$ . For each such  $\tau$ ,  $\mu([\tau]) = 2^{-f(n)}$ . Thus  $\mu(U_n) \leq 2^{f(n)-n} \cdot 2^{-f(n)} = 2^{-n}$ . Hence  $(U_n)_{n \geq 0}$  is a primitive recursive test which succeeds on  $X$ , and  $X$  is not Martin-Löf BP random.

(2)  $\Rightarrow$  (1): Suppose  $X$  is not BP Martin-Löf random. By Proposition 2.3,  $X \in \bigcap_n U_n$  for some weak primitive recursive test  $(U_n)_{n \geq 0}$ . As usual,  $U_n = [G_n]$ , where  $G_n = \{\tau_{1,n}, \dots, \tau_{k(n),n}\}$  is primitive recursively computable. Let  $f(n)$  be the length of the strings  $\tau_{i,n}$ .

We define a martingale  $d$  as follows. For  $n = 1$ , and given  $U_1 = [\tau_{1,1}] \cup \dots \cup [\tau_{k,1}]$ , we let  $d(\tau_{i,1}) = \frac{2^{f(1)}}{k}$  for  $i = 1, \dots, k$ . If  $\tau \in \{0, 1\}^{f(1)} \setminus \{\tau_{1,1}, \dots, \tau_{k,1}\}$ , then we let  $d(\tau) = 0$ .

Since  $\mu(U_1) \leq \frac{1}{2}$ , it follows that  $k \leq 2^{f(1)-1}$  and therefore  $d(\tau_{i,1}) \geq 2$  for each  $i$ . Moreover,  $\sum_{\tau \in \{0,1\}^{f(1)}} d(\tau) = k \cdot \frac{2^{f(1)}}{k} = 2^{f(1)}$ . Now work backwards using the martingale equation  $d(\sigma) = \frac{1}{2}(d(\sigma \frown 0) + d(\sigma \frown 1))$  to define  $d(\sigma)$  for all  $\sigma$  of length  $\leq f(1)$ . It follows by induction that for all  $j \leq f(1)$ ,  $\sum_{\tau \in \{0,1\}^j} d(\tau) = 2^j$  so that, in particular,  $d(\emptyset) = 1$ .

Now suppose that  $n \geq 1$  and we have defined  $d(\tau)$  for all  $\tau$  with  $|\tau| \leq f(n)$  so that  $d(\tau) \geq 2^n$  for all  $\tau \in G_n$ . We need to extend  $d$  to strings of length  $\leq f(n+1)$ . For  $\sigma$  of length  $f(n)$ , we will define  $d(\sigma\tau)$ , where  $\sigma\tau = \sigma \frown \tau$ , for all  $\tau$  of length  $f(n+1) - f(n)$ . If  $d(\sigma) = 0$ , then we simply let  $d(\sigma\tau) = 0$  for all  $\tau$ . Now fix  $\sigma \in G_n$  with  $d(\sigma) \geq 2^n$  and consider  $H = \{\tau : \sigma\tau \in G_{n+1}\}$ . Since  $(U_n)_n$  is a weak test,  $\mu([H]) \leq \frac{1}{2}$ . Thus we may proceed as in the first case where  $n = 1$  to define a martingale  $m$  such that  $m(\emptyset) = 1$  and  $m(\tau) \geq 2$  for all  $\tau \in H$ . Now extend the definition of  $d$  to the strings extending  $\sigma$  by defining  $d(\sigma\tau) = d(\sigma) \cdot m(\tau)$ . Since  $d(\sigma) \geq 2^n$  and, for  $\tau \in H$ ,  $m(\tau) \geq 2$ , it follows that for  $\sigma\tau \in G_{n+1}$ ,  $d(\sigma\tau) \geq 2^{n+1}$ . It is easy to see that this extension obeys the martingale equality, since, for any  $\tau$ ,

$$d(\sigma\tau) = d(\sigma) \cdot m(\tau) = d(\sigma) \cdot \frac{1}{2}(m(\tau \frown 0) + m(\tau \frown 1)) = \frac{1}{2} \cdot (d(\sigma\tau \frown 0) + d(\sigma\tau \frown 1)).$$

Since  $X \in \bigcap_n U_n$ , it follows that  $d(X \upharpoonright f(n)) \geq 2^n$  for each  $n$  and hence  $d$  succeeds primitive recursively on  $X$ .

It is easy to see that this defines a primitive recursive procedure to compute  $d(\sigma)$ . The first step is to compute  $f(n)$  for  $n \leq |\sigma|$  until we find  $n$  so that  $|\sigma| \leq f(n)$ . Then we consider all extensions  $\tau$  of  $\sigma$  of length  $f(n)$ . We follow the procedure outlined above to compute  $d(\sigma \upharpoonright f(i))$  for  $i \leq n$ , and, hence, compute  $d(\tau)$  for all extensions  $\tau$  of  $\sigma$  of length  $f(n)$ . Finally we backtrack using the martingale inequality to compute  $d(\sigma)$  from the values of such  $d(\tau)$ . Thus  $d$  is a primitive recursive martingale and  $X$  is not martingale BP random.  $\square$

The above proofs are similar to those that have already appeared in the literature. For example Downey, Griffiths and Reid [16] proved the equivalence of analogues of (1) and (3) of Theorem 2.7 in the setting of Kurtz randomness, and their proof could be modified to prove the equivalence of (1) and (3) in Theorem 2.7. Similarly, Bienvenu and Merkle [6] gave a proof the equivalence of parts (1) and (2) in the setting of Kurtz randomness and their proof can be modified to work in our setting.

Given Theorems 2.7 and 2.8, we define  $X \in \{0,1\}^\omega$  to be BP *random* if and only if  $X$  is Martin-Löf BP random. Since every BP test is also a Kurtz test and a computable test, it follows that all Kurtz random and all computably random reals are BP random.

It is clear that no primitive recursive set can be BP random. It was shown by Jockusch [17] that Kurtz random sets are *immune*, that is, they do not include any c.e. subsets. Here is a version of that result for BP randomness.

**Proposition 2.9.** If  $X$  is BP random, and  $f$  is an increasing primitive recursive function, then  $X$  does not contain the range of  $f$ .

*Proof.* Suppose for the contrapositive that  $X$  contains the range of  $f$ . Let  $G_n = \{\sigma \in \{0,1\}^{f(n)} : (\forall i < n)(\sigma(f(i)) = 1)\}$ , and  $U_n = [G_n]$ . It is clear that  $\mu([U_n]) = 2^{-n}$  so that  $(U_n)_{n \geq 0}$  is a primitive recursive test. But then  $X$  belongs to each  $U_n$  so  $X$  is not BP random.  $\square$

**Theorem 2.10.** There is a recursive real which is BP random.

*Proof.* Let  $(M_e, f_e)$  enumerate all pairs of primitive recursive functions, where  $M_e : \{0, 1\}^* \rightarrow \{0, 1\}^*$  and  $f_e : \mathbb{N} \rightarrow \mathbb{N}$  is strictly increasing. Let  $C_e$  denote  $C_{M_e}$ . We define a recursive real  $X$  such that, for any  $e$ , there is some  $c$  such that  $C_e(X \upharpoonright f_e(c)) > f_e(c) - c - 1$ . Thus no primitive recursive machine can primitive recursively compress  $X$ , so  $X$  is BP random.

The definition of  $X$  is in stages, with  $X$  defined as the union of a sequence  $\emptyset = \tau_0 \subseteq \tau_1 \subseteq \dots$ . Let  $n_k = |\tau_k|$ . We set  $\tau_0 = \emptyset$  and  $n_0 = 0$ .

At stage  $k + 1$ , we let  $c = n_k$ , and  $n_{k+1} = f_k(n_k)$ . We are looking for an extension  $\tau = \tau_{k+1}$  of  $\tau_k$  of length  $f_k(c)$  which does not equal  $M_k(\sigma)$  for any  $\sigma$  of length  $\leq f_k(c) - c - 1$ . There are  $2^{f_k(c)-c}$  different extensions of  $\tau_k$  of length  $f_k(c)$ , but only  $2^{f_k(c)-c} - 1$  strings of length  $\leq f_k(c) - c - 1$ . Hence such a string  $\tau$  exists and we may compute it primitive recursively (and in a space efficient manner) by the following algorithm. Enumerate all possible values for  $\tau \in \{0, 1\}^{n_{k+1}}$  starting with  $0^{n_{k+1}}$ . For each value  $\tau$ , compute  $M(\sigma)$  for all strings  $\sigma$  of length  $\leq f_k(c) - c - 1$ . If the values  $M(\sigma)$  are all distinct from the candidate value for  $\tau$ , then output that  $\tau$ .  $\square$

Next we show that BP random reals satisfy the following analogue of Ville's Theorem.

**Theorem 2.11.** Let  $X \in \{0, 1\}^\omega$  be BP random and let  $g$  be a primitive recursive increasing function. Then the sequence  $(X(g(0)), X(g(1)), X(g(2)), \dots)$  is also BP random.

*Proof.* Let  $Y(n) = X(g(n))$ , and suppose that  $Y$  is not BP random. Let  $(U_n)_{n \geq 0}$  be a primitive recursive test such that  $Y \in \bigcap_n U_n$ . As usual, we may assume that  $U_n = [G_n]$  where  $G_n$  is a subset of  $\{0, 1\}^{f(n)}$  for some primitive recursive function  $f$ . In other words, each  $\tau \in G_n$  has length  $f(n)$ .

We define a primitive recursive test  $(V_n)_n$  for  $X$  by letting  $V_n \subset \{0, 1\}^{g(f(n))}$  be

$$V_n = \{X : (X(g(0)), X(g(1)), \dots, X(g(f(n)))) \in U_n\}.$$

It is easy to see that  $\mu(U_n) = \mu(V_n)$ , and  $(V_n)_{n \geq 1}$  is a primitive recursive test. Also,  $X \in \bigcap_{n \geq 1} V_n$  which violates the assumption that  $X$  is BP random. This contradicts the assumption that  $Y$  is BP random.  $\square$

**2.3. Prefix-free primitive recursive.** Kolmogorov complexity and randomness are usually studied for prefix-free machines. By convention, primitive recursive functions are total, and thus not prefix-free. Nonetheless, we can consider partial primitive recursive functions by adding a new symbol,  $\infty$ , for divergence. The usual definitions of primitive recursive functions can be readily modified to allow the primitive function to output the special symbol  $\infty$  to indicate the function diverges. With this convention, a primitive recursive function  $M$  is *prefix-free* provided that there do not exist distinct strings  $\sigma \sqsubset \tau$  such that both  $M(\sigma) \neq \infty$  and  $M(\tau) \neq \infty$ .

**Definition 2.12.** A sequence  $X \in \{0, 1\}^\omega$  is *prefix-free BP random* if there does not exist a prefix-free primitive recursive function  $M$  and a primitive recursive function  $f$  such that  $C_M(X \upharpoonright f(c)) \leq f(c) - c$  for all  $c$ .

**Proposition 2.13.** A real  $X$  is BP random if and only if it is prefix-free BP random.

*Proof.* Certainly if  $X$  is BP random then it is prefix-free BP random. So suppose that  $X$  is not BP random. By Theorem 2.7, there is a primitive recursive test  $\{U_c : c \in \mathbb{N}\}$  such that  $X \in \bigcap_c U_c$ . We may assume (by replacing  $U_c$  with  $U_{2c}$  if necessary) that in fact  $\mu(U_c) \leq 2^{-2c}$ . As usual,  $U_n$  is defined in terms of a primitive recursive  $G_n = \{\tau_{1,n}, \dots, \tau_{k(n),n}\}$  where each  $\tau_{i,n}$  has length  $f(n)$ , so  $U_n = [G_n]$ . As in the proof of Theorem 2.7, we may assume that  $f(c+1) - (c+1) > f(c) - c$ . From  $|U_c| < 2^{-2c}$ , we have  $k(c)2^{-f(c)} \leq 2^{-2c}$ . Hence  $k(c) \leq 2^{f(c)-2c}$ . It follows that  $f(c) - c \geq c$  for all  $c$ .

We need to define a prefix-free primitive recursive function  $M$  such that  $C_M(X \upharpoonright f(c)) \leq f(c) - c$  for all  $c$ . We start by defining  $M(\sigma)$  for  $\sigma$  of length  $< f(1)$ . Let  $\sigma_{1,1}, \sigma_{2,1}, \dots, \sigma_{k(1),1}$  be the lexicographically first  $k(1)$  strings of length  $f(1) - 1$ , and define  $M(\sigma_i) = \tau_{i,1}$ . And, let  $M(\sigma) = \infty$  for all other strings of length  $< f(1)$ . Note that  $M(\sigma) \neq \infty$  for at most half of the strings of length  $f(1) - 1$ , since  $k(1) \leq 2^{f(1)-2}$ .

Now suppose that we have defined the partial function  $M$ , in a prefix-free way, for strings of length  $\leq f(c-1) - c + 1$ , so that the following hold: First,  $M$  is injective (where it converges). Second, for each  $n < c$  and  $\tau_{i,n} \in G_n$ , there is a unique  $\sigma$  of length  $f(n)$  such that  $M(\sigma) = \tau_{i,n}$ . Third,  $M(\tau)$  is undefined for all other strings  $\sigma$  of length  $\leq f(c-1) - c + 1$ . Fourth, for each  $n < c$ ,  $M(\sigma) \neq \infty$  for at most fraction  $2^{-n}$  of the strings of length  $f(n) - n$ . We want to extend  $M$  by setting the values of  $M(\sigma)$  for all  $\sigma$  of length  $< f(c) - c$ . For  $f(c-1) - c + 1 < |\sigma| < f(c) - c$ , set  $M(\sigma) = \infty$ . In order to define  $M(\sigma)$  for  $|\sigma| = f(c) - c$ , enumerate the lexicographically first strings  $\sigma_{1,c}, \sigma_{2,c}, \dots, \sigma_{k(c),c}$  of length  $f(c) - c$  which do not extend any  $\nu$  for which  $|\nu| < f(c) - c$  and  $M(\nu) \neq \infty$ . Once these are enumerated, define  $M(\sigma_{i,c}) = \tau_{i,c}$ . To see that it is possible to enumerate such strings, first note that  $k(c) \leq 2^{-c} \cdot 2^{f(c)-c}$ . Furthermore, for each  $n < c$ , at most a fraction  $2^{-n}$  of the strings  $\nu$  in  $\{0,1\}^{f(n)-n}$  have  $M(\nu) \neq \infty$ . Since  $\sum_{n=0}^{c-1} 2^{-n} < 1 - 2^{-c+1}$ , there are more than  $k(c)$  strings. In fact, there are at least  $2k(c)$  available to enumerate. Thus  $M$  is well-defined.

By assumption,  $X \in U_c$  for every  $c$ , so  $X \upharpoonright f(c) = \tau_{i,c}$  for some  $i$  and hence  $M(\sigma_{i,c}) = \tau_{i,c} = X \upharpoonright f(c)$ . Since  $|\sigma_{i,c}| = f(c) - c$ , we have  $C_M(X \upharpoonright f(c)) = f(c) - c$  as desired.

It remains to check that  $M$  is indeed a primitive recursive function. Recall that  $f(c) - c \geq c$ . Thus, given a string  $\sigma$ , we only need to check  $c \leq |\sigma|$  to see whether  $m = f(c) - c$  for some  $c$ . If not, then  $M(\sigma) = \infty$ . If so, we use the above construction of  $M$  to determine whether  $\sigma = \sigma_{i,c}$  where  $M(\sigma_{i,c}) = \tau_{i,c}$ . This is clearly a primitive recursive procedure for computing  $M(\sigma)$ .  $\square$

**2.4. Statistical tests.** It is interesting to see to what extent the BP random sets are statistically random. We begin with a positive result.

**Theorem 2.14.** Let  $X$  be a BP random set. For any increasing primitive recursive function  $f$  and any  $\epsilon > 0$ ,

$$\left| \frac{\text{card}(X \cap [[f(n)]])}{f(n)} - \frac{1}{2} \right| \leq \epsilon$$

for infinitely many  $n$ .

*Proof.* This follows from the law of large numbers (Chernoff's Lemma [24, p. 61]). For a finite string  $\sigma$ , let  $\text{card}(\sigma)$  denote  $\text{card}(\{i : \sigma(i) = 1\})$ . For  $n \in \mathbb{N}$  and any real  $\epsilon > 0$ , let

$$S_{n,\epsilon} = \left\{ \sigma \in \{0,1\}^n : \left| \frac{\text{card}(\sigma)}{n} - \frac{1}{2} \right| > \epsilon \right\}.$$

Chernoff's Lemma states that  $\mu([S_{n,\epsilon}]) \leq 2e^{-\epsilon^2 n/6}$ .

Fix  $\epsilon > 0$  and let  $m = \lceil 1/\epsilon \rceil$ . Let  $V_n = S_{6m^2(n+1) \ln 2, \frac{1}{m}}$ . It follows by an easy calculation from Chernoff's Lemma that  $\mu(V_n) \leq 2^{-n}$ . Finally, let  $U_n = V_{f(n)}$ . Since  $f$  is increasing,  $f(n) > n$  and hence  $\mu(U_n) \leq 2^{-f(n)} \leq 2^{-n}$  for all  $n$ . We claim that  $(U_n)_n$  is in fact a primitive recursive test. This is because  $U_n = [G_n]$ , where  $G_n$  is a set of strings of length  $6m^2 f(n)$  and the membership of  $\tau$  in  $G_n$  is easily computable by counting the numbers of 0's and 1's in  $\tau$ .

Since  $X$  is BP random, it follows that  $X \notin U_n$  for at least one  $n$ . In fact, by considering the tests  $(U_{n+i})_n$  for each  $i \in \mathbb{N}$ , we see that  $A \notin U_n$  for infinitely many  $n$ .  $\square$

**Corollary 2.15.** For any BP random set  $X$ , if  $\lim_n \text{card}(X \cap [[n]])/n$  exists, then it equals  $1/2$ .

On the other hand, BP random sets do not have to be stochastic. Note that this is also the case for Kurtz random sets.

**Theorem 2.16.** There is a computable BP random set  $X$  such that  $\lim_n \text{card}(X \cap [[n]])/n$  does not exist.

*Proof.* To construct such a set  $X$ , modify the proof of Theorem 2.10 by adding long strings of 0's and long strings of 1's (in alternation) after satisfying each requirement. Then we can make the density go arbitrarily low and then arbitrarily high infinitely often.  $\square$

**2.5. Relative randomness.** We now turn to the concept of *relative* primitive recursive randomness, namely, BP randomness relative to an oracle  $Y$ . Let  $Y = (Y(0), Y(1), Y(2), \dots)$  be a real. The set of functions which are primitive recursive relative to  $Y$  can be defined in two equivalent ways. The first definition uses the characterization of the primitive recursive functions as the closure of a set of base functions under composition and primitive recursions. Namely, let the *base* primitive recursive functions consist of the constant functions  $f(\vec{x}) = c$  for  $c \in \mathbb{N}$ , the successor function  $f(x) = x + 1$ , and the projection functions  $f(x_1, \dots, x_k) = x_i$ . Then a function is *primitive recursive relative to  $Y$*  if it can be obtained from the base functions plus the function  $Y$  under the usual primitive recursion operations of composition and primitive recursion.

The second definition is in terms of Turing machines with runtimes bounded by primitive recursive functions. Namely, a Turing machine, denoted  $M^Z$ , is given oracle access to the values of  $Z$  in the usual way by giving it a query state and an oracle query/answer tape. An oracle Turing machine  $M^Z$  is called a *primitive recursive oracle machine* provided there is a primitive recursive function  $g$ , such that for all  $x$  and all reals  $Z$ ,  $M^Z(x)$  has runtime bounded by  $g(x)$ . A function  $f^Z$  is a *primitive recursive oracle function* provided there is some primitive recursive oracle machine  $M^Z$  such that for all  $x$  and all reals  $Z$ ,  $f^Z(x) = M^Z(x)$ . Then, for a fixed oracle  $Y$ , a function  $f(x)$  is *primitive recursive relative to  $Y$*  if and only if there is some primitive recursive oracle Turing machine  $M^Z$  such that  $f(x) = M^Y(x)$ .

The equivalence of the two definitions for primitive recursive relative to  $Y$  is well-known, and depends on the fact that  $Y$  is a 0/1-valued function and thus is majorized by a primitive recursive function.

We shall define a real  $X = (X(0), X(1), \dots)$  to be Martin-Löf BP random relative to  $Y$ , Kolmogorov BP random relative to  $Y$ , and martingale BP random relative to  $Y$

by replacing (some of) the primitive recursive functions in the definitions of Martin-Löf BP random, Kolmogorov BP random, and martingale BP random by primitive recursive functions relative to  $Y$ , respectively.

2.5.1. *The compressibility definition.* Let  $C_M^Y(\tau)$  be the length  $|\sigma|$  of the shortest string  $\sigma$  such that  $M^Y(\sigma) = \tau$ , that is, the length of the shortest  $M^Y$ -description of  $\tau$ .

**Definition 2.17.** An infinite sequence  $X$  is *primitive recursively compressed relative to  $Y$*  if there exist a primitive recursive oracle machine  $M$  and a primitive recursive function  $f$  such that, for every  $c$ ,  $C_M^Y(X \upharpoonright f(c)) \leq f(c) - c$ . An infinite sequence  $X$  is *Kolmogorov bounded primitive recursively random relative to  $Y$*  (for short, *Kolmogorov BP random relative to  $Y$* ) if it cannot be primitive recursively compressed relative to  $Y$ .

If in the above definition, the condition that  $f$  is primitive recursive is replaced by  $f$  is primitive recursive relative to  $Y$ , then the set of reals which are Kolmogorov bounded primitively random relative to  $Y$  would not change. This is because we can always find a primitive recursive function  $f'$  (not relative to  $Y$ ) that dominates  $f$ , and then use the construction mentioned immediately after the proof of Theorem 2.7.

2.5.2. *The measure definition.* A *primitive recursive oracle test* is given by a function  $g^Z$  which is primitive recursive relative to  $Z$ , and a primitive recursive function  $f$  such that, for all reals  $Z$ , (1)  $g^Z(n)$  codes a finite set, denoted  $G_n^Z$  of strings, with  $G_n^Z \subseteq \{0, 1\}^{f(n)}$ , (2)  $U_n^Z = [G_n^Z]$  is a clopen set, and (3)  $\mu(U_n^Z) < 2^{-n}$ . If these conditions hold, then, for a particular real  $Y$ , we also refer to the sequences  $(G_n^Y)_n$  and  $(U_n^Y)_n$  as being primitive recursive oracle tests.

**Definition 2.18.** A real  $X$  is *Martin-Löf BP random relative to a real  $Y$*  if, for any primitive recursive oracle test  $(U_n^Y)_n$ ,  $X \notin U_n^Y$  for some  $n$ .

As before, the definition of Martin-Löf BP random relative to a real  $Y$  would be unchanged if the function  $f$  were allowed to be primitive recursive relative to  $Y$  instead of just primitive recursive.

As a simple example, suppose that  $X$  is itself primitive recursive and let  $G_n^Z = [X \upharpoonright n]$  for all  $n$  and  $Z$ , so that  $f(n) = n$ . Then  $(G_n^Z)_{n \geq 0}$  is a primitive recursive oracle test, and  $X \in [G_n^Z]$  for all  $n$ . Thus  $X$  is not Martin-Löf BP random relative to *any*  $Z$ .

On the other hand, suppose that  $X$  is Martin-Löf BP random and let  $Y$  be primitive recursive. We claim then that  $X$  is Martin-Löf BP random relative to  $Y$ . If not, let  $(U_n^Y)_n$  be a primitive recursive oracle test such that  $X \in U_n^Y$  for all  $n$ . Then in fact  $(U_n^Y)_n$  is a primitive recursive test since  $Y$  is primitive recursive. And  $X$  fails this test, contradicting the assumption that  $X$  is Martin-Löf BP random.

The notion of a *weak* test may also be relativized to say that, as before, for each  $\tau \in G_n^Z$ ,  $\mu(U_{n+1}^Z \cap [\tau]) \leq \frac{1}{2}\mu([\tau])$ . The proof of Proposition 2.2 relativizes so that we have the following proposition.

**Proposition 2.19.**  $X$  is Martin-Löf BP random relative to  $Y$  if and only if it passes every weak primitive recursive oracle test relative to the oracle  $Y$ .  $\square$

2.5.3. *The martingale definition.* A primitive recursive oracle function  $d^Z$  is a *primitive recursive oracle martingale* if  $d^Z$  is a martingale for all reals  $Z$ .

**Definition 2.20.** A real  $X$  is *martingale BP random relative to  $Y$*  if there is no primitive recursive oracle martingale  $d^Y$  which succeeds primitive recursively on  $X$ .

It is straightforward to check that the proof of Theorem 2.7 relativizes to prove the following.

**Theorem 2.21.** The following are equivalent for  $X, Y \in \{0, 1\}^\omega$ .

- (1)  $X$  is Kolmogorov BP random relative to  $Y$ .
- (2)  $X$  is Martin-Löf BP random relative to  $Y$ .
- (3)  $X$  is martingale BP random relative to  $Y$ .

When these conditions hold, we say that  $X$  is BP *random relative to  $Y$* .

2.5.4. *van Lambalgen's theorem.* We now discuss van Lambalgen's theorem for relative BP randomness. We first prove that one direction of van Lambalgen's theorem holds in this setting, and then give a counterexample for the other direction. Recall that if  $A, B \subseteq \mathbb{N}$ , then  $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$ .

**Theorem 2.22.** If  $A \oplus B$  is BP random, then  $A$  is BP random relative to  $B$  and  $B$  is BP random relative to  $A$ .

*Proof.* Suppose  $B$  is not BP random relative to  $A$ , and there is a primitive recursive oracle test  $(U_n^A)_n$  such that  $B \in \bigcap_n U_n^A$ . The test  $(U_n^A)_n$  is given by a primitive recursive oracle function  $g^Z$  so that, for all reals  $Z$ ,  $g^Z(n)$  codes a set  $G_n^Z \subset \{0, 1\}^*$  so that  $U_n^A = [G_n^Z]$ . The runtime of  $g^Z(n)$  is primitive recursively bounded; in particular, there is a primitive recursive function  $\ell(n)$  so that, for all  $Z$  and all  $n$ ,  $g^Z(n)$  only queries values of  $Z(i)$  for  $i < \ell(n)$ . In addition,  $G_n^Z \subset \{0, 1\}^{\leq \ell(n)}$  for all  $n$ . For  $z = \{0, 1\}^*$  with  $|z| \geq \ell(n)$ , we can thus unambiguously define  $g^z(n)$  to equal the value of  $g^Z(n)$  when run on any  $Z \sqsupset z$ . In this situation, we write  $G^z(n)$  for the set coded by  $g^z(n)$ , so  $G_n^z = G_n^Z$  for any  $Z \sqsupset z$ . We define a primitive recursive test  $(V_n)_n$  by letting  $V_n = [H_n]$  where

$$H(n) = \{a \oplus b : a, b \in \{0, 1\}^{\ell(n)} \text{ and } b \in [G_n^a]\}.$$

Clearly,  $H_n$  is primitive recursive since  $b \in [G_n^a]$  holds iff some prefix of  $b$  is in  $G_n^a$ . Furthermore, since for each  $a \in \{0, 1\}^{\ell(n)}$ ,  $\mu([G_n^a]) \leq 2^{-n}$ , we have  $\mu(V_n) \leq 2^{-n}$ . In addition, since  $B \in U_n^A$  for all  $n$ , we also have  $A \oplus B$  is in  $V_n$  for all  $n$ . Therefore  $A \oplus B$  is not BP random.

It follows by symmetry that if  $A \oplus B$  is BP random, then  $A$  is BP random relative to  $B$ .  $\square$

Next we give a counterexample to the converse of Theorem 2.22.

**Theorem 2.23.** There are reals  $A$  and  $B$  such that  $A$  is BP random relative to  $B$ , and  $B$  is BP random relative to  $A$ , but  $A \oplus B$  is not BP random.

*Proof.* We construct the reals  $A$  and  $B$  in stages. The stages will be controlled by a fast growing increasing function  $h(i)$ : at the end of stage  $i$ , the values of  $A(n)$  and  $B(n)$  for  $n < h(i)$  will have been set. Furthermore, these values will satisfy:

1. For  $h(2j) \leq n < h(2j + 1)$ , we have  $B(n) = 0$ .
2. For  $h(2j + 1) \leq n < h(2j + 2)$ , we have  $A(n) = 0$ .



Define the sequences  $A_i, B_i \in \{0, 1\}^{h(i)}$  to consist of the values of  $A(n)$  and  $B(n)$  that have been set by the end of the  $i$ -th stage. In the end, we set  $A = \lim_i A_i$  and  $B = \lim_i B_i$ . Conditions 1 and 2 clearly imply that  $A \oplus B$  is not BP random.

Let  $(M_j^Z, f_j)_{j \geq 0}$  enumerate all pairs such that  $M_e^Z : \{0, 1\}^* \mapsto \{0, 1\}^*$  is a primitive recursive oracle function, and  $f_j : \mathbb{N} \rightarrow \mathbb{N}$  is a primitive recursive function with  $f_j(c) > c$  for all  $c$ . Stage  $2j + 1$ , which sets the values of  $A(n)$  and  $B(n)$  for  $h(2j) \leq n < h(2j + 1)$ , will ensure that  $A$  is not primitive recursively compressed by  $(M_j^B, f_j)$ . Stage  $2j + 2$  will act similarly to ensure that  $B$  is not primitive recursively compressed by  $(M_j^A, f_j)$ .

We describe an odd stage  $2j + 1$ . The value of  $h(2j)$  has already been set; we must define  $h(2j + 1)$  and the values of  $A(n)$  for  $h(2j) \leq n < h(2j + 1)$ . Define  $X$  to be the infinite sequence consisting of  $B_{2j}$  followed by all 0's. Let  $c = h(2j) + 1$  and compute  $f_j(c) > c$ . There are  $2^{f_j(c) - h(2j)}$  many strings of the form  $A_{2j}u$  with  $|A_{2j}u| = f_j(c)$ . On the other hand, there are at most  $2^{f_j(c) - h(2j)} - 1$  strings of the form  $M_j^X(v)$  with  $|v| \leq f_j(c) - h(2j) - 1 = f_j(c) - c$ . Fix a value for  $u$  so that  $|A_{2j}u| = f_j(c)$  and  $u$  is not equal to  $M_j^X(v)$  for any  $v \in \{0, 1\}^{\leq f_j(c) - c}$ .

Now, set  $h(2j + 1)$  to be equal to the least value  $\geq f_j(c)$  such that, for every  $v \in \{0, 1\}^{\leq f_j(c) - c}$ ,  $M_j^X(v)$  only queries oracle values  $X(m)$  for  $m < h(2j + 1)$ . This ensures that  $M_j^X(v) = M_j^B(v)$  for all such  $v$ 's. Set  $A_{2j+1} = A_{2j}u0^{h(2j+1) - f_j(c)}$ , so  $|A_{2j+1}| = h(2j + 1)$ . Further set  $B_{2j+1} = B_{2j}0^{h(2j+1) - h(2j)}$ . By construction,  $C_{M_j^B}(A \upharpoonright f_j(c)) > f_j(c) - c$ , so  $A$  is not primitive recursively compressed by  $(M_j^B, f_j)$ .

That completes the description of the even stages. The odd stages are defined similarly.  $\square$

Kihara and Miyabe [18] have obtained a similar result for Kurtz random reals, using the same basic idea for the proof.

### 3. POLYNOMIAL-SPACE BOUNDED RANDOMNESS

In this section, we modify the definitions of BP randomness to define a notion of bounded randomness relative to polynomial space functions, called ‘‘BPS randomness’’. It is important to note that this is a notion of bounded randomness, similar to the bounded primitive recursive randomness developed above.

The usual notion of PSPACE randomness is part of the well-developed study of resource-bounded measure and randomness due to Lutz, Schnorr, and Ambos-Spies and Mayordomo, see [25, 3, 31]. It will follow from our definitions that PSPACE randomness implies BPS randomness. The two notions are certainly different, as PSPACE random sets satisfy the law of large numbers, whereas BPS random sets, like BP random sets, satisfy only a limited form of this law, see, for example, Theorem 3.13.

We begin by defining three equivalent notions of BPS randomness, analogous to the three versions of BP randomness.

First, however, we need to define polynomial space functions. A PSPACE *predicate*, or *set*, is one for which membership can be decided by a Turing machine which uses space bounded by a polynomial  $p(n)$  of the length  $n$  of its input. A polynomial space computable *function* is a function  $\{0, 1\}^* \rightarrow \{0, 1\}^*$  which can be computed by a Turing machine  $M$  with a read-only input tape, work tapes, and a write-only output tape such that the space that  $M$  uses on its work tapes is polynomially bounded by the length  $n$  of its input. It

is required that  $M$  halts for all possible inputs, thus the runtime of  $M(x)$  is bounded by  $2^{p(n)}$  for some polynomial  $p$ , where  $n = |x|$  is the length of the input  $x$ . If in addition, the length of output of  $M$  is polynomially bounded, then we say  $M$  computes a function in FPSPACE. That is, an FPSPACE function  $f$  is computable in polynomial space, and there is a polynomial  $p(n)$  such that  $|f(x)| \leq p(|x|)$  for all inputs  $x$ . On the other hand, if we do not bound the length of  $f(x)$ , so that  $|f(x)|$  is bounded only by  $2^{p(n)}$  for some polynomial  $p$ , then we call  $f$  an FPSPACE<sup>+</sup> function. However, we are primarily interested in FPSPACE functions.

We will use FPSPACE functions for defining all three of Martin-Löf, Kolmogorov, and martingale BPS randomness. (So, FPSPACE<sup>+</sup> functions are not needed for the definitions.) All three types of definitions use PTIME functions  $f : \{1\}^* \rightarrow \{1\}^*$  to bound the lengths of strings. The importance of these PTIME functions lies solely in the fact that their growth rate is bounded by a polynomial; in other words, in the fact that  $|f(1^n)| \leq p(n)$  for some polynomial  $p$ . In fact, all our definitions could equivalently use just functions of the form  $f(1^n) = 1^{n^c+c}$  for  $c \in \mathbb{N}$ . Similarly, the definitions could equivalently use FPSPACE functions  $f : 1^* \rightarrow \{0, 1\}^*$ , as their growth rate is also bounded by a polynomial.

**Definition 3.1.** A PSPACE test  $(U_n)_{n \geq 0}$  is specified by a pair of functions  $(G, f)$  such that  $G : \{1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}$  is an FPSPACE-function and  $f : \{1\}^* \rightarrow \{1\}^*$  is a strictly length increasing PTIME function such that for each  $n$ ,

$$G_n = \{\tau \in \{0, 1\}^{\leq |f(1^n)|} : G(1^n, \tau) = 1\}$$

is a set of strings of length  $\leq |f(1^n)|$  such that  $U_n = [G_n]$  is a clopen set with measure  $\leq 2^{-n}$ .

As is easy to check, PSPACE tests could be equivalently defined with a FPSPACE<sup>+</sup> function  $g$  instead of the FPSPACE function  $G$ , by defining  $G_n$  to be equal to the set coded by the (potentially exponentially long) string  $g(1^n)$ .

The above definition gave  $G_n$  as the domain of a parameterized FPSPACE function  $G$ . An alternate, and equivalent, definition is to define  $G_n$  as the range of a parameterized PSPACE function  $\widehat{G}$ :

**Proposition 3.2.** The sequence  $(G_n)_n$  and  $f$  satisfy the conditions for the definition of a PSPACE test iff there is an FPSPACE function  $\widehat{G} : \{1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^*$  such that

$$G_n = \{\sigma \in \{0, 1\}^{\leq |f(1^n)|} : \text{for some } \tau \in \{0, 1\}^{\leq |f(1^n)|}, \widehat{G}(1^n, \tau) = \sigma\}.$$

Therefore, we will also refer to  $(\widehat{G}, f)$  as being a PSPACE test.

A *weak* PSPACE test  $(U_n)_{n \geq 0}$  is a PSPACE test as above with the additional property that for each  $n$  and  $\sigma_{i,n} \in G_n$ ,  $\mu(U_{n+1} \cap [\sigma_{i,n}]) \leq \frac{1}{2}\mu([\sigma_{i,n}])$ .

**Definition 3.3.** An infinite sequence  $X$  is *Martin-Löf BPS random* if  $X$  passes every PSPACE test.  $X$  is *weakly Martin-Löf BPS random* if  $X$  passes every weak PSPACE test.

**Definition 3.4.** An infinite sequence  $X$  is *Kolmogorov BPS random* if there do not exist an FPSPACE function  $M : \{0, 1\}^* \rightarrow \{0, 1\}^*$  and a PTIME function  $f : \{1\}^* \rightarrow \{1\}^*$  such that, for every  $n \in \mathbb{N}$ ,  $C_M(X \upharpoonright |f(1^n)|) \leq |f(1^n)| - n$ .

For the next definition, we take members of  $\mathbb{Q} \cap [0, \infty)$  as being coded as ternary strings  $\sigma 2\tau$  where  $\sigma.\tau$  is the binary expansion of a rational number in  $\mathbb{Q} \cap [0, \infty)$ .

**Definition 3.5.** An FPSPACE martingale  $d : \{0, 1\}^* \rightarrow \mathbb{Q} \cap [0, \infty)$  *succeeds* on  $X$  if there is a PTIME function  $f : \{1\}^* \rightarrow \{1\}^*$  such that, for all  $n$ ,  $d(X \upharpoonright m) \geq 2^n$  where  $m = |f(1^n)|$ . An infinite sequence  $X$  is *martingale BPS random* if no FPSPACE martingale succeeds on  $X$ .

We have the following analogue of Proposition 2.6:

**Proposition 3.6.** The following are equivalent:

- (1):  $X$  is martingale BPS random.
- (2): There do not exist an FPSPACE martingale  $d$  and a PTIME function  $f$  such that, for every  $n$ , there exists  $m \leq f(n)$  such that  $d(X \upharpoonright m) \geq 2^n$ .
- (3): There do not exist an FPSPACE martingale  $d$  and a PTIME function  $f$  such that  $d(X \upharpoonright m) \geq 2^n$  for all  $n$  and all  $m \geq f(n)$ .

*Proof.* As before, the implications (2)  $\Rightarrow$  (1) and (1)  $\Rightarrow$  (3) are immediate. Proving the implication (3)  $\Rightarrow$  (2) requires a refined version of the argument used in the proof of Proposition 2.6. As before, we use the idea of a savings account, but now the intuition is that once  $d(\tau) \geq 4^{n+1}$ , then one half of the working capital is transferred to a savings account. So, loosely speaking, every time the capital is *quadrupled*, one half of it is moved into the savings account. Suppose  $d$  and  $f$  are as in (2). We formally define a martingale  $d'$  as follows. Given  $\tau \in \{0, 1\}^*$ , define  $\ell_n^\tau$  to equal the least value  $\leq |\tau|$ , if any, such that  $d(\tau \upharpoonright \ell_n^\tau) \geq 4^{n+1}$ . Let  $\tau_n$  equal  $\tau \upharpoonright \ell_n^\tau$ , and let  $n(\tau)$  be the least value of  $n$  for which  $\ell_n^\tau$  is undefined. Note that  $\ell_n^\tau$ ,  $\tau_n$ , and  $n(\tau)$  are in FPSPACE, since  $d$  is in FPSPACE. Now define

$$d'(\tau) = \frac{d(\tau)}{2^{n(\tau)}} + \sum_{i=0}^{n(\tau)-1} \frac{d(\tau_i)}{2^{i+1}}.$$

Once again, the summation represents the capital placed in the savings account, and  $d(\tau)/2^{n(\tau)}$  is the remaining working capital. As before, it is easy to verify that  $d'$  is a martingale. Also, by the definition,  $d'(\tau)$  is in FPSPACE.

Since  $d(\tau_i) \geq 4^{i+1}$ , we have  $d(\tau) \geq \sum_{i=1}^{n(\tau)-1} 2^{i+1} > 2^{n(\tau)}$ . Let  $f'$  be the PTIME function  $f'(1^n) = f(1^{2^n})$ . Suppose  $X \in \{0, 1\}^\omega$  and  $m \geq |f'(1^n)|$ . Since  $m \geq |f(1^{2^n})|$ , there is some  $m' \leq m$  such that  $d(X \upharpoonright m') \geq 2^{2^n} = 4^n$ . Therefore,  $n(X \upharpoonright m) \geq n$ , so  $d'(X \upharpoonright m) \geq 2^n$ . This establishes that the properties of (3) holds for  $d'$  and  $f'$ .  $\square$

By suitably modifying the proofs of Theorem 2.7 and 2.8, we can prove the equivalence of these three versions of BPS randomness. As we shall see, the modifications are straightforward in the case of Theorem 2.7, but more substantial for Theorem 2.8.

As a preliminary step, the next lemma states that, for Martin-Löf BPS randomness, the strings in the sets  $G_n$  of a PSPACE test can be required to all be the same length.

**Lemma 3.7.** For any PSPACE test  $(G, f)$ , there is an FPSPACE function  $G'$  such that  $(G', f)$  is a PSPACE test such that, for all  $n$ ,  $G'_n \subset \{0, 1\}^{f(n)}$ .

*Proof.* We are given that  $G_n = \{\sigma \in \{0, 1\}^{\leq f(n)} : G(1^n, \sigma) = 1\}$ . It will suffice to define an FPSPACE function  $G'$  so that  $G'_n = \{\sigma \in \{0, 1\}^{f(n)} : G'(1^n, \sigma) = 1\}$  satisfies  $[G_n] = [G'_n]$  for all  $n$ . This is simply done: define  $G'(1^n, \sigma) = 1$  to hold exactly when  $|\sigma| = n$  and there is some  $\tau \sqsubset \sigma$  such that  $G(1^n, \tau) = 1$ . Clearly  $[G_n] = [G'_n]$ , and  $G'$  is in FPSPACE since  $G$  is.  $\square$

**Definition 3.8.** A function  $M : \{0, 1\}^* \rightarrow \{0, 1\}^*$  is said to be an FPSPACE BPS *quick process machine* if  $M$  is in FPSPACE and is a quick process machine with order function  $h$  such that there is a PTIME function  $g : \{1\}^* \rightarrow \{1\}^*$  such that  $g(1^n) = 1^{h(n)}$  for all  $n$ . An infinite sequence  $X$  is *quick process BPS random* if there do not exist an FPSPACE BPS quick process machine  $M$  and a PTIME function  $f : \{1\}^* \rightarrow \{1\}^*$  such that, for every  $n \in \mathbb{N}$ ,  $C_M(X \upharpoonright |f(1^n)|) \leq |f(1^n)| - n$ .

**Theorem 3.9.** The following are equivalent for  $X \in \{0, 1\}^\omega$ .

- (1)  $X$  is Martin-Löf BPS random.
- (2)  $X$  is Kolmogorov BPS random.
- (3)  $X$  is quick process BPS random.

*Proof.* The proof is almost identical to the proof of Theorem 2.7, but noting that all the constructions preserve the property of being computable in FPSPACE, instead of being primitive recursive. As before, (2)  $\Rightarrow$  (3) is immediate. The proof of (1)  $\Rightarrow$  (2) is identical to the proof of the same case for Theorem 2.7. The only difference is that now  $G_c$  must be the range of a parameterized FPSPACE function  $\widehat{G}$  with  $G_c = \{\sigma : \sigma = \widehat{G}(1^c, \tau), \tau \in \{0, 1\}^{|f(1^n)|}\}$ . Referring back to the definition of  $G_c$  in the proof Theorem 2.7 this is immediately seen to hold, as  $M$  is in FPSPACE.

The proof of (3)  $\Rightarrow$  (1) is also identical to the corresponding proof of Theorem 2.7; however, it is required to show that the function  $M$  is in FPSPACE. In fact, the construction given in the proof Theorem 2.7 gives a polynomial space algorithm for  $M$ . Recall that the values of  $M(\sigma)$  for  $f(c) - c < |\sigma| \leq f(c+1) - c - 1$  were computed by expressing  $\sigma$  in the form  $\sigma = M(\nu) \frown \rho$ , if possible, where  $f(c-1) - c + 1 < |\nu| \leq f(c) - c$ , and then enumerating  $H_\nu$  in lexicographic order. Since  $G_c$  is given as the parameterized domain of the FPSPACE function  $\widehat{G}$ , it is certainly possible to enumerate  $G_c$  and hence  $H_\nu$  using only polynomial space. It is also necessary to find the value  $\nu$ , if any, such that  $M(\nu) \sqsubset \sigma$ . This is done by calculating  $M(\nu)$  for all  $\nu \in \{0, 1\}^{|f(c)|-c}$ .

In other words,  $M(\sigma)$  is computed in polynomial space by a recursive procedure that needs to compute values of  $M(\nu)$  with  $f(c-1) - c + 1 < |\nu| \leq f(c) - c$ . The depth of the recursion is equal to  $c$ ; hence, the overall computation of  $M(\sigma)$  can be carried out in polynomial space.  $\square$

Our next goal is to show that  $X$  is martingale BPS random if and only if  $X$  is weakly Martin-Löf BPS random. The following lemma about martingales is a version of Kraft's Inequality and will be needed to help us prove this result. Two strings  $\sigma, \tau \in \{0, 1\}^*$  are said to be *incompatible* if they are distinct and neither one is a prefix of the other.

**Lemma 3.10** (Kraft's Inequality). For any martingale  $d$ , any  $\sigma \in \{0, 1\}^*$  and any finite set  $H$  of pairwise incompatible extensions of  $\sigma$ ,

$$\sum_{\tau \in H} d(\tau) \cdot 2^{-|\tau|} \leq d(\sigma) \cdot 2^{-|\sigma|}.$$

*Proof.* Let  $n = \max\{|\tau| : \tau \in H\}$  and note that by the martingale condition,

$$\sum \{d(\rho) : |\rho| = n \text{ \& } \sigma \sqsubset \rho\} = d(\sigma) \cdot 2^{n-|\sigma|}.$$

For each  $\tau \in H$ , let  $G(\tau) = \{\rho \in \{0, 1\}^n : \tau \sqsubset \rho\}$ . Then by the martingale condition as above

$$\sum_{\rho \in G(\tau)} d(\rho) = d(\tau) \cdot 2^{n-|\tau|}.$$

Thus

$$\begin{aligned} \sum_{\tau \in H} d(\tau) \cdot 2^{-|\tau|} &= \sum_{\tau \in H} \sum_{\rho \in G(\tau)} d(\rho) \cdot 2^{-n} \\ &\leq \sum \{d(\rho) \cdot 2^{-n} : |\rho| = n \text{ \& } \sigma \sqsubset \rho\} = d(\sigma) \cdot 2^{-|\sigma|}. \end{aligned}$$

□

We next prove the analogue of Theorem 2.8 that martingale BPS randomness is equivalent to the other forms of BPS randomness. The proof of Theorem 2.8 needs considerable reworking however. This is primarily because the proof of Theorem 2.8 depended on the equivalence of weak Martin-Löf BPS randomness and Martin-Löf BPS randomness. Unfortunately, the proof of Proposition 2.3 cannot be readily modified to apply to BPS randomness: the difficulty is that that proof defined a function  $h$  so that  $h(n+1) = \ell(h(n)) + 1$ , but  $h$  may not be in PTIME even if  $\ell$  is. Nonetheless, it does follow from the next theorem that any PSPACE test can be converted into a weak PSPACE test.

**Theorem 3.11.** The following are equivalent for  $X \in \{0, 1\}^\omega$ .

- (1)  $X$  is Martin-Löf BPS random.
- (2)  $X$  is weakly Martin-Löf BPS random.
- (3)  $X$  is martingale BPS random.

*Proof.* Clearly (1) implies (2).

(2)  $\Rightarrow$  (3): Suppose that (3) fails and  $d$  is an FSPACE martingale which succeeds on  $X$  with a PTIME function  $f$  such that, for all  $n$ ,  $d(X \upharpoonright |f(1^n)|) \geq 2^n$ . Unlike the (2)  $\Rightarrow$  (1) case of Theorem 2.8, we must construct a weak PSPACE test that  $X$  fails instead of just a PSPACE test. Define  $G : \{1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}$  by letting  $G(1^n, \sigma) = 1$  iff

$$|\sigma| \leq |f(1^n)| \text{ \& } d(\sigma) \geq 2^n \text{ \& } (\forall i < |\sigma|)(d(\sigma \upharpoonright i) < 2^n).$$

Note that, by the martingale equality, if  $G(1^n, \sigma) = 1$ , then  $d(\sigma) < 2^{n+1}$ . Clearly  $G$  is an FSPACE function. Thus

$$G_n = \{\sigma : |\sigma| \leq |f(1^n)| \text{ \& } d(\sigma) \geq 2^n \text{ \& } (\forall i < |\sigma|)(d(\sigma \upharpoonright i) < 2^n)\}.$$

Set  $U_n = [G_n]$ ; we have  $X \in U_n$  for all  $n$  by the assumption. By Lemma 3.10 with  $\sigma = \emptyset$ ,  $\sum_{\tau \in G_n} d(\tau) \cdot 2^{-|\tau|} \leq 1$ . Since for all  $\tau \in G_n$ ,  $d(\tau) \geq 2^n$ , it follows that

$$\mu([U_n]) = \sum_{\tau \in G_n} 2^{-|\tau|} \leq 2^{-n}.$$

Now let  $V_n = U_{2n}$  for all  $n \geq 0$  so that  $(V_n)_{n \geq 0}$  is a PSPACE test that  $X$  fails. We claim that  $(V_n)_{n \geq 0}$  is a weak test. We have  $\mu(V_n) = \mu(U_{2n}) \leq 2^{-2n}$ . For  $\sigma \in V_n$ , let  $H(\sigma) = \{\tau : \sigma \sqsubset \tau \text{ \& } \tau \in V_{n+1}\}$ . By Lemma 3.10,

$$\sum_{\tau \in H(\sigma)} d(\tau) \cdot 2^{-|\tau|} \leq d(\sigma) \cdot 2^{-|\sigma|}.$$

Since  $d(\sigma) < 2^{2n+1}$  and for each  $\tau \in H(\sigma)$ ,  $d(\tau) \geq 2^{2n+2}$ , we obtain

$$\sum_{\tau \in H(\sigma)} 2^{2n+2} \cdot 2^{-|\tau|} \leq 2^{2n+1} \cdot 2^{-|\sigma|}.$$

Dividing by  $2^{2n+2}$ , we obtain

$$\mu([H(\sigma)]) = \sum_{\tau \in H(\sigma)} 2^{-|\tau|} \leq \frac{1}{2} \cdot 2^{-|\sigma|} = \frac{1}{2} \mu([\sigma]).$$

Thus the sequence  $V_n$  is a weak PSPACE test as desired. Thus if  $X$  is not martingale BPS random, then  $X$  is not weakly Martin-Löf BPS random.

(3)  $\Rightarrow$  (1): This is the most difficult case to prove, since we have not (yet) established the analogue of Proposition 2.3 for BPS randomness. Let  $\{U_n\}_{n \geq 0}$  be a Martin-Löf BPS test which  $X$  fails, with  $U_n = [G_n]$  where  $G_n$ 's are as usual given by an FPSPACE function  $G$ . In particular, each  $U_n$  has measure  $\mu(U_n) \leq 2^{-n}$ , and w.l.o.g. every  $\sigma \in G_n$  has length equal to  $\ell(n)$  for some PTIME function  $\ell$  such that  $\ell(n) \geq i$ .

We will define martingales  $d_i$  for  $i = 1, 2, \dots$ , and then define the overall martingale which succeeds against  $X$  as  $d = \sum_i 2^{-i} d_i$ . The martingale  $d_i$  is defined using the set  $U_{3i-1}$ . That is to say, we use  $U_j$  only for  $j = 3i - 1$ .

**Definition 3.12.** First consider  $i = 1$ . The martingale  $d_1$  will be defined from  $G_2$  so as to satisfy the following conditions:

1.  $d_1(\emptyset) = 1$ .
2.  $d_1(\sigma) = 4$  for each  $\sigma \in G_2$ .
3.  $d_1(\sigma 0) = d_1(\sigma 1) = d_1(\sigma)$  for all  $\sigma$  of length  $|\sigma| \geq \ell(2)$ .

The point of condition 3. is that the value of  $d_1(\sigma)$  “settles down” to a constant value which depends only on the first  $\ell(2)$  symbols of  $\sigma$ . Property 2. can be forced to hold since  $\mu(U_2) \leq 1/4$ , and hence the martingale properties allow the value of  $d_1(\emptyset)$  to be quadrupled for  $\sigma \in G_2$ .

It is straightforward to define an FPSPACE function  $d_1$  satisfying these three conditions 1-3. One way to do this is to enumerate the members of  $G_2$ : since  $G_2 \subset \{0, 1\}^{\ell(2)}$  and  $\mu([G_2]) \leq 1/4$ , we have  $|G_2| \leq 2^{\ell(2)-2}$ . Let  $H_2$  be the lexicographically first  $2^{\ell(2)-2} - |G_2|$  many strings of length  $\ell(2)$  which are not in  $G_2$ . Then define  $d_1$  as follows: For  $\sigma = \ell(2)$ ,  $d_1(\sigma)$  is set equal to 4 if  $\sigma \in G_2 \cup H_2$ , and set equal to zero otherwise. For longer strings  $\sigma$ ,  $d_1(\sigma) = d_1(\sigma \upharpoonright \ell(2))$ . For shorter strings  $\sigma$ ,  $d_1(\sigma)$  is computed using Kraft's inequality and counting the number of  $\tau \sqsupset \sigma$  with  $\tau \in G_2 \cup H_2$ .

**Definition 3.13.** Now consider  $i > 1$ . The martingale  $d_i$  is defined so as to satisfy

- 1.i  $d_i(\sigma) = 1$  for every  $\sigma$  of length  $|\sigma| < i$ . In particular,  $d_i(\sigma)$  is constant for small  $\sigma$ 's.
- 2.i  $d_i(\sigma) = 2^{2i}$  for every  $\sigma \in G_{3i-1}$ .
- 3.i  $d_i(\sigma 0) = d_i(\sigma 1) = d_i(\sigma)$  for all  $\sigma$  of length  $|\sigma| \geq \ell(3i - 1)$ .

The idea for making property 2.i hold with  $i > 1$  is similar to the  $i = 1$  construction for  $d_1$ . Property 2.i is possible since

- (a) there are  $2^{i-1}$  strings  $\sigma$  of length  $i - 1$ , and for these  $\sigma$ 's,  $d_i(\sigma) = 1$  and
- (b) we have  $\mu(U_{3i-1}) \leq 2^{-(3i-1)}$  so that  $|G_{3i-1}| \leq 2^{\ell(3i-1)-(3i-1)}$ .

For any  $\sigma$  of length  $i - 1$ ,

$$\mu(U_{3i-1} \cap [\sigma]) / \mu([\sigma]) \leq \mu(U_{3i-1}) / \mu([\sigma]) \leq 2^{-(3i-1)} / 2^{-(i-1)} = 2^{-2i}.$$

Thus  $d_i$ 's value of 1 for  $\sigma$  may be multiplied by the factor  $2^{2i}$  for each  $\tau \in U_{3i}$  which extends  $\sigma$ . That is, define  $H_{3i-1}$  to contain, for each  $\sigma \in \{0, 1\}^{i-1}$ , the lexicographically first  $k(\sigma)$  many extensions of  $\sigma$  in  $([\sigma] \cap [\{0, 1\}^{\ell(3i-1)}]) \setminus G_{3i-1}$  where  $k(\sigma)$  equals  $2^{2i} - |G_{3i-1} \cap [\sigma]|$ . Then define  $d_i(\sigma)$  as follows: For  $\sigma \in \{0, 1\}^{\ell(3i-1)}$ , set  $d_i(\sigma)$  equal to  $2^{2i}$  if  $\sigma \in G_{3i-1} \cup H_{3i-1}$ ,

and equal to zero otherwise. For shorter strings  $\sigma$ , set  $d_i$  using Kraft's inequality; by construction, this satisfies 1. For longer strings  $\sigma$ ,  $d_i(\sigma) = d_i(\sigma \upharpoonright \ell(3i-1))$ . By construction, each  $d_i(\sigma)$  can be computed in FPSPACE, uniformly in  $i$  and  $\sigma$ , by enumerating the strings in  $G_{3i-1}$ .

We finally claim that  $d(\sigma)$  can be computed in polynomial space. For this,

$$d(\sigma) = \sum_{i \geq 0} 2^{-i} d_i(\sigma) = \sum_{0 \leq i \leq |\sigma|} 2^{-i} d_i(\sigma) + \sum_{i > |\sigma|} 2^{-i} d_i(\sigma).$$

The finite summation  $\sum_{0 \leq i \leq |\sigma|} d_i(\sigma)$  can be computed in polynomial space by just evaluating each  $d_i(\sigma)$ . The other, infinite, summation  $\sum_{i > |\sigma|} d_i(\sigma)$  is equal to  $2^{-|\sigma|}$  since  $d_i(\sigma) = 1$  for  $|\sigma| < i$ . Thus  $d(\sigma)$  can be computed in polynomial space.

Since  $X \upharpoonright \ell(3i-1) \in U_{3i-1}$ ,

$$d(X \upharpoonright \ell(3i-1)) \geq 2^{-i} d_i(X \upharpoonright \ell(3i-1)) = 2^{-i} \cdot 2^{2i} = 2^i,$$

and  $d$  is an FPSPACE martingale which succeeds against  $X$ . This concludes the proof of Theorem 3.11.  $\square$

We have the following analogue of Proposition 2.9. This is a version of bi-immunity for BPS random sets. The notion of immunity is well-studied in the resource-bounded randomness community. For example, it is known that any ptime random set is ptime bi-immune; see Ambos-Spies and Mayordomo [3].

**Proposition 3.14.** Let  $X$  be BPS random. Suppose  $h : \{1\}^* \rightarrow \{0, 1\}^*$  is an FPSPACE function such that  $f(n) = |h(1^n)|$  is an increasing function. Then  $X$  does not contain the range of  $f$ .

*Proof.* Argue as in the proof of Proposition 2.9.  $\square$

Next, analogously to Theorem 2.10, we construct a BPS random real which is not too complex. A DSPACE( $t(n)$ ) real is an infinite sequence  $X \in \{0, 1\}^\omega$  which is computable by a DSPACE( $t(n)$ ) function (which is 0/1 valued, since  $X$  is 0/1 valued).

**Theorem 3.15.** Let  $\epsilon > 0$ . Then there is a DSPACE( $2^{\epsilon(\log n)^2}$ ) real which is Martin-Löf BPS random.

*Proof.* We modify the proof of Theorem 2.10 as follows. Let  $(M_e, f_e)$  enumerate all pairs of functions such that  $M_e : \{0, 1\}^* \rightarrow \{0, 1\}^*$  is computable in space  $\max\{2, n\}^{\max\{2, \log e\}}$  and  $f_e : \{1\}^* \rightarrow \{1\}^*$  is computable in time  $\max\{2, n\}^{\max\{2, \log e\}}$ . Every possible BPS compression will be found in this list, since every computable function has infinitely many indices. We may also assume without loss of generality that  $f_e(i) \geq i$  for all  $e$  and  $i$ . Let  $C_e$  denote  $C_{M_e}$ . We will define a real  $X$  such that, for every  $e$ , there is some  $c$  such that  $C_e(X \upharpoonright f_e(c)) > f_e(c) - c - 1$ . It follows that no BPS machine can BPS compress  $X$ , and hence  $X$  is BPS random.

The definition of  $X$  proceeds as in the proof of Theorem 2.10. That is, the definition of  $X$  is in stages, and  $X$  is the union of a sequence  $\emptyset = \tau_0 \subseteq \tau_1 \subseteq \dots$ . Let  $n_k = |\tau_k|$ . Initially  $\tau_0 = \emptyset$  and thus  $n_0 = 0$ .

At stage  $k+1$ , we let  $c = n_k$ , and  $n_{k+1} = f_k(n_k)$ . Then we find an extension  $\tau = \tau_{k+1}$  of  $\tau_k$  of length  $f_k(c)$  which does not equal  $M_k(\sigma)$  for any  $\sigma$  of length  $\leq f_k(c) - c - 1$ . Observe that there are  $2^{f_k(c)-c}$  different extensions of  $\tau_k$  of length  $f_k(c)$ , but there are only  $2^{f_k(c)-c-1}$  strings of length  $\leq f_k(c) - c - 1$ . Hence such a string  $\tau$  exists and we may compute it as

follows. Start with the guess that  $\tau = 0^{n_{k+1}}$  and successively compute  $M_k(\sigma)$ , and compare it with  $\tau$ , for every  $\sigma$  of length  $\leq f_k(c) - c - 1$ . If we find a match, then change  $\tau$  to  $0^{n_{k+1}-1}1$  and continue incrementing  $\tau$  until we find the desired string.

It remains to be seen that  $X$  is in fact computable in space  $n^{\epsilon \log n}$ . To compute  $X \upharpoonright n$  from  $n$ , first compute the sequence

$$n_0 = 0, n_1 = f_0(n_0), \dots, n_{k+1} = f_k(n_k)$$

until  $n_{k+1} \geq n$ . Now  $X \upharpoonright n$  will be an initial segment of  $\tau_{k+1}$ , which will have length  $n_{k+1} = f_k(n_k) \leq n^{\epsilon \log n}$ , for sufficiently large  $n$ . This last inequality is because  $f_k(m)$  is computable in time  $n^{\log k}$  and we have  $n_k \leq n$  and also  $k \leq n$  since  $f_e(i) \geq i$ . Thus  $M_k(\sigma)$  can be computed in space  $\leq n^{\epsilon \log n}$  for all strings  $\sigma$  of length  $\leq n_{k+1} - n_k - 1$ . It follows that the entire computation up through stage  $k+1$ , needed to compute  $X \upharpoonright n$ , can be done in space  $n^{\epsilon \log n}$ .  $\square$

We also have the following analogue of Theorem 2.11.

**Proposition 3.16.** Let  $X \in \{0,1\}^\omega$  be BPS random and let  $h : \{1\}^* \rightarrow \{0,1\}^*$  be computable in FPSPACE. Suppose that  $g(n) = |h(1^n)|$  is an increasing function. Then the sequence  $(X(g(0)), X(g(1)), X(g(2)), \dots)$  is also BPS random.

*Proof.* The construction from the proof of Theorem 2.11 still applies; we only need to verify that  $V_n$  is an FPSPACE test. To see this, just note that  $g(f(n))$  is computable in space bounded by  $p(n)$  for some polynomial  $p$ . From this,  $V_n$  is a PSPACE test.  $\square$

We can also give a prefix-free characterization of BPS randomness, completely analogous to the case of BP randomness.

**Definition 3.17.** A sequence  $X \in \{0,1\}^\omega$  is *prefix-free BPS random* if there does not exist a prefix-free FPSPACE function  $M$  and a PTIME function  $f$  such that  $C_M(X \upharpoonright f(c)) \leq f(c) - c$  for all  $c$ .

**Proposition 3.18.** A real  $X$  is BPS random if and only if it is prefix-free BPS random.

*Proof.* The proof of Theorem 2.13 applies as is. The only change needed is to verify that the algorithm for  $M$  can be carried out in polynomial space. This is proved using the same kind of argument used for the proof of the (3)  $\Rightarrow$  (1) case of Theorem 3.9.  $\square$

The theorems on statistical tests also carry over to the setting of BPS random sets:

**Theorem 3.19.** Let  $X$  be a BPS random set. For any increasing primitive recursive function  $f$  and any  $\epsilon > 0$ ,

$$\left| \frac{\text{card}(X \cap [[f(n)]])}{f(n)} - \frac{1}{2} \right| \leq \epsilon$$

for infinitely many  $n$ .

*Proof.* The proof is identical to the proof of Theorem 2.14. The fact that  $U_n$  is a PSPACE test follows from the fact that membership in  $S_{n,\epsilon}$  is decidable in PSPACE for any fixed  $\epsilon$ .  $\square$



**Corollary 3.20.** For any BPS random set  $X$ , if  $\lim_n \text{card}(A \cap [[n]])/n$  exists, then it equals  $1/2$ .

**Theorem 3.21.** Let  $\epsilon > 0$ . Then there exists a BPS random set  $A$  in  $\text{DSPACE}(2^{\epsilon(\log n)^2})$  such that  $\lim_n \text{card}(A \cap [[n]])/n$  does not exist.

*Proof.* To construct such a set  $A$ , modify the proof of Theorem 3.15 by adding long strings of 0's and long strings of 1's (in alternation) after satisfying each requirement. Then we can make the density go arbitrarily low and then arbitrarily high infinitely often.  $\square$

It is an immediate consequence of (the proof of) Theorem 3.21, that there are BPS random reals which are not PSPACE random in the sense of Lutz [26]. To see this, note that part 4 of example 2.10 of Ambos-Spies and Mayordomo [3] shows that PSPACE random reals cannot have “exponential gaps”. For the same reason, it follows that there are BPS random reals which are not PSR1-/PSR2-random in the sense of Ko [19], as Ko showed that the relative frequencies of 0's and 1's in a PSR1-/PSR2-random real quickly converge to  $1/2$ .

The constructions of Section 2.5 carry over straightforwardly to BPS randomness. Since we only use relative computation relative to 0/1 valued functions, the definitions of the various forms of relative BPS randomness are completely straightforward. It is then easy to prove the equivalence of the the Kolmogorov definition, the Martin-Löf definition, and the martingale definition of relative randomness by the same constructions as used for Theorems 3.9 and 3.11. We leave the details to the reader. One direction of van Lambalgen's theorem also follows by the same proof as Theorem 2.22.

**Theorem 3.22.** For any reals  $A$  and  $B$ , if  $A$  is BPS random relative to  $B$  and  $B$  is BPS random relative to  $A$ , then  $A \oplus B$  is BPS random.

On the other hand, the sets  $A$  and  $B$  constructed in the proof of Theorem 2.23 are certainly BPS random relative to each other, and clearly  $A \oplus B$  is not BPS random. Thus,

**Theorem 3.23.** There are reals  $A$  and  $B$  such that  $A$  is BPS (even, BP) random relative to  $B$ , and  $B$  is BPS (even, BP) random relative to  $A$ , but  $A \oplus B$  is not BPS random.

#### 4. CONCLUSIONS AND FUTURE RESEARCH

In this paper, we defined a robust notion of primitive recursive and PSPACE bounded random reals in that both definitions could be framed in all three versions of algorithmically random reals via measure, Kolmogorov complexity, or martingales. We view the work of this paper as a possible model for defining algorithmically random reals relative to several other classes of sub-computable functions. In future work, we will define similar notions of bounded random reals for other classes of sub-computable functions such as elementary, on-line, or exponential space.

Lutz [25, 26] proved a wide range of properties of PSPACE randomness. It would be interesting to understand which of the properties of PSPACE randomness established in [25, 26] apply also to BPS randomness. Lutz showed that  $P$  has measure 0 in EXPTIME. This means that there is a single EXPTIME martingale which succeeds on every set in  $P$ . It is an interesting question whether there can be a BPS martingale which succeeds on every set in  $P$ .

Kihara and Miyabe [18] defined a notion of uniform relativization for Kurtz randomness for which van Lambalgen's theorem holds. It would be interesting to investigate whether their uniform relativization can be adapted to the setting of BP and BPS randomness.

A theory of algorithmic randomness for trees and effectively closed sets was developed in a series of papers by Barmpalias, Brodhead, Cenzer, et al. [4, 5]. One can adapt our definitions of primitive recursive bounded randomness to define similar notions of bounded random trees and effectively closed sets for various classes of sub-computable functions. This will appear in future papers.

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