

StConn: *st*-Connectivity Tautologies

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Abstract—We describe a software package for generating the StConn (*st*-Connectivity) tautologies, as a set of unsatisfiable CNFs.

Index Terms—*st*-connectivity, satisfiability, conjunctive normal form formula, software

I. INTRODUCTION

The *st*-Connectivity tautologies, StConn, were introduced in [1]. They express the principle that if two curves cross each other, they must intersect at some point. Specifically, consider a grid graph where the vertices are a $d \times n$ rectangular array and the potential edges join neighboring vertices horizontally or vertically. A red path starts in the bottom left corner of the grid and ends at the upper right corner. Likewise, a blue path starts in the upper left corner and ends at the lower right corner. The StConn principle states that the paths must intersect at some vertex. (See Figures 1 and 2.)

See [1] for motivation on why the StConn principle might be difficult, and a proof that it has resolution refutations of width $d + O(1)$ and size $O(n2^d)$. This is polynomial size if d is constant.

II. VARIABLES AND CLAUSES

Edges e in the graph are unordered pairs of vertices. Horizontal edges have the form $\{(i, j), (i, j+1)\}$ where $0 \leq i < n$ and $0 \leq j \leq d$. Vertical edges have the form $\{(i, j), (i, j+1)\}$ where $0 \leq i \leq n$ and $0 \leq j < d$. For each edge e there are variables r_e and b_e indicating whether a red edge or a blue edge is present at that position.

A vertex is a *corner* vertex if it is one of the four corner vertices $(0, 0)$, $(n, 0)$, $(0, d)$, or (n, d) .

The clauses state the following:

- At each non-corner vertex (i, j) there are either 0 or 2 red edges incident to that vertex. Likewise there are either 0 or 2 incident blue edges.
- Each corner vertex $(0, 0)$ and (n, d) has exactly one incident red edge and no incident blue edges.
- Each corner vertex $(0, d)$ and $(n, 0)$ has exactly one incident blue edge and no incident red edges.
- For each pair of edges e and f that have a common vertex, i.e., that have $e \cap f \neq \emptyset$, the clause $\bar{r}_e \vee \bar{b}_f$ is present. These clauses enforce the condition that the red and blue path have no common vertices.

The unsatisfiability of these clauses expresses the validity of the *st*-connectivity principle.

III. USAGE

The StConn software is a C++ program. It is invoked by the command line as

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% StConn <d> <n>
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where $\langle n \rangle$ is the length n of the graph (from left-to-right in the figures) and $\langle d \rangle$ is the width d of the grid graph (shown horizontally in the figures).

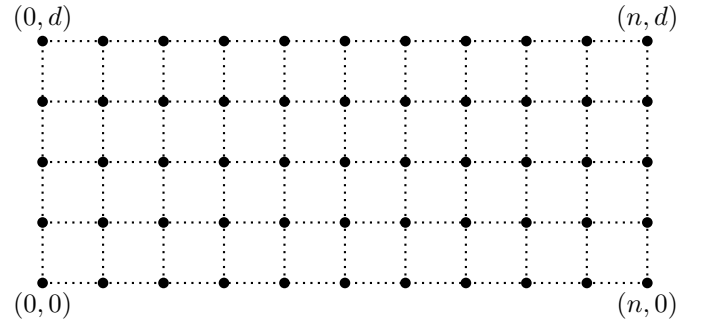


Fig. 1. The grid graph is a $d \times n$ rectangular array of vertices. The value d is the “width” of the graph; n is the length. The potential edges in the graph join vertices that are horizontally or vertically adjacent, as indicated by the dotted lines. Each edge, if present, will be colored either red or green.

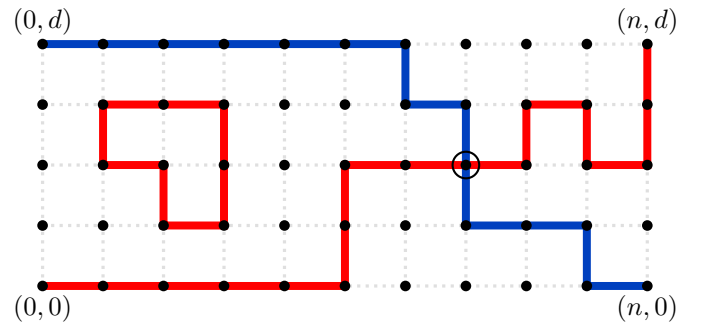


Fig. 2. The red and blue paths connect diagonally opposite corners and intersect at the vertex $(7, 2)$. The StConn tautology states that if the paths are well-formed then they must intersect at some vertex. The StConn clauses do not rule out the possibility of having extraneous loops, as shown by the red loop in the figure.

IV. ACKNOWLEDGEMENTS

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REFERENCES

- [1] S. R. Buss, “Polynomial-size Frege and resolution proofs of *st*-connectivity and Hex tautologies,” *Theoretical Computer Science*, vol. 1-3, pp. 35–52, 2006.