Nelson's Formalist Approach to Feasible Arithmetic and the Foundations of Mathematics

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Nelson's formalist philosophy: A radical form of constructivism

Platonists believe in the full, independent existence our usual mathematical constructs, including integers, reals, the powerset of the reals, even abstract sets.

Most **constructivists**, in the spirit of D. Hilbert, accept use of the completed infinity of integers, the use of primitive recursive functions, etc.

Ed Nelson adopted a strict **formalist** philosophy, coupled with an **ultra-constructivism** that does not accept the totality of exponentiation and other primitive recursive functions.

I. Doubt About the Integers?

How could one doubt the integers? Even without believing in the integers as “physical” entities, one surely should believe in them as a set of mental constructs that have definite properties.

For example: **Do there exist odd perfect integers?**

Whether there exists an odd perfect integer should be a definite property of the integers. That is, they exist, or do not exist, independently of the successes or failures human efforts in doing mathematics.

In contrast, reasonable people might agree to doubt the relevance or the meaning of the continuum hypotheses CH or GCH. There could be multiple, equally compelling concepts of “set” and thus no reason to believe that CH or GCH have any independent meaning as a platonic truth (or platonic falsity).

On Doubting the Integers

Nelson [PA, p.1]:
The reason for mistrusting the induction principle is that it involves an impredicative concept of number. It is not correct to argue that induction only involves the numbers from 0 to n; the property of n being established may be a formula with bound variables that are thought of as ranging over all numbers. **That is, the induction principle assumes that the natural number system is given.** A number is conceived to be an object satisfying every inductive formula; for a particular inductive formula, therefore, the bound variables are conceived to range over objects satisfying every inductive formula, including the one in question.
(emphasis added)

Nelson [A, p232]:

> If we attempt to justify induction from below, saying that a number is one of 0, $S0$, $SS0$, $SSS0$, and so forth, we make the category error of conflating the genetic with the formal. If we attempt to justify induction from above, saying that numbers by definition satisfy every inductive formula, then we are using an impredicate concept of number.

Nelson [Confessions, p.3]:

> Everything in creation is contingent; every created thing is dependent on the will of the Creator for its being. If numbers are uncreated, they are divine — this we reject. If numbers are created, they are contingent — this we find absurd. What other possibility is there? Simply that numbers do not exist — not until human beings make them.

Nelson [M&F, p.7]:

> I must relate how I lost my faith in Pythagorean numbers. One morning at the 1976 Summer Meeting of the American Mathematical Society in Toronto, I woke early. As I lay meditating about numbers, I felt the momentary overwhelming presence of one who convicted me of arrogance for my belief in the real existence of an infinite world of numbers, leaving me like an infant in my crib reduced to counting on my fingers. Now I live in a world in which there are no numbers save those that human beings on occasion construct.

II. Formalism

Nelson [Confessions, pp.6-7]:

> Formalism denies the relevance of truth to mathematics.
> [...]
> In mathematics, reality lies in the symbolic expressions themselves, not in any abstract entity they are thought to denote.
> [...]
> What is real in mathematics is the notation, not an imagined denotation.

Ed, however, also had doubts about the unrestrained use of formalism. As he realized, a formalist deals with symbolic expressions. Symbolic expressions can code integers, and can be coded by integers (Gödel coding).

Thus an unrestrained use of formalism is tantamount to accepting the set of integers as a completed infinity.

Both strands of thought came together in Ed Nelson's mathematical development of *predicative theories of arithmetic.*

III. Nelson's Predicative Arithmetic

Hilbert's program: Hilbert suggested that as a first step before considering "truth" or "semantics", one should consider "syntax" of proofs and the "consistency" of theories.

Hilbert's program for establishing consistency was foiled by Gödel's incompleteness theorems already at the level of formal theories of arithmetic. Nelson took this failure at face value and doubted even the consistency of commonly used theories of the integers!

Welcome fact: Even if one does not buy into formalism or ultra-constructivism, there is still interesting mathematics to do with weak systems of consistency strength much weaker than Peano arithmetic.

Predicative arithmetic: a weak form of arithmetic which does not make platonic assumptions about the existence of a completed infinity of integers.

Base Language and Axioms of Predicative Arithmetic

First-Order Logic: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\forall$, $\exists$, $=$

Fucntion Symbols: 0, S (successor), $+$ (addition), $\cdot$ (multiplication).

Robinson's Theory Q:

$Sx \neq 0$	$x + 0 = x$
$Sx = Sy \rightarrow x = y$	$x + Sy = S(x + y)$
$x \neq 0 \rightarrow (\exists y)(Sy = x)$	$x \cdot 0 = 0$
	$x \cdot Sy = x \cdot y + x.$

Q is very weak, but still subject to Gödel's incompleteness theorem. Q does not even prove $0 + x = x$ or $x + y = y + x$.

Q does *not* include any induction axioms. It is a very weak subtheory of Peano arithmetic (PA).

Extensions by Definition

The theory Q can be conservatively extended by adding new symbols defined in terms of old ones. For example:

Inequality Predicate:

$$x \leq y \quad \Leftrightarrow \quad (\exists z)(x + z = y)$$

Predecessor Function:

$$P(x) = y \quad \Leftrightarrow \quad S(y) = x \vee (x = 0 \wedge y = 0).$$

Extension by definitions gives a conservative extension of Q, so they can be used freely. (But totality of predecessor needs to be established!)

The consistency of assuming $0 + x = x$

Def'n A formula $\phi(x)$ is inductive provided it has been proved that

$$\phi(0) \;\wedge\; \forall x(\phi(x) \to \phi(Sx)).$$

Theorem: (Solovay[unpub.],Nelson[PA]) Suppose $\phi(x)$ is inductive. Let

$$\begin{array}{lll} \phi^1(x) & \text{be} & \forall y(y \le x \to \phi(y)). \\ \phi^2(x) & \text{be} & \forall y(\phi^1(y) \to \phi^1(y+x)). \\ \phi^3(x) & \text{be} & \forall y(\phi^2(y) \to \phi^2(y \cdot x)). \end{array}$$

Then, $\phi^3(x)$ defines an initial segment of the integers (called a "cut") which contains 0, and is closed under S, $+$ and $\cdot$. The property φ is valid on this initial segment.

Thm: Let $\psi(x)$ be the formula $0 + x = x$. Then, $\psi(x)$ is inductive.

Pf: (a) $0 + 0 = 0$ holds by an axiom of Q.
(b) Suppose $0 + x = x$. Then $0 + Sx = S(0 + x) = Sx$. $\square$.

Thus $\{x : \psi^3(x)\}$ is an initial segment of integers that satisfies

$$Q_2 := Q + \forall x(0 + x = x).$$

This has provided an "interpretation" of Q_2 in Q.

Semantic viewpoint: starting with a mass of 'integers' that satisfy Q, we have found an initial segment of "integers" that also satisfies $\forall x(0+x = x)$.

Syntactic viewpoint: if Q is consistent, then Q_2 is consistent. Using 'relativization', any Q_2-proof can be transformed into a Q-proof. By relativization is meant restricting attention to integers that satisfy $\psi^3(x)$.

Definition of Predicative:

Def'n [Nelson, PA] A theory $T \supset Q$ is predicative if it is interpretable in Q. This include allows multiple uses of extension by definition and of interpretation with initial segments obtained from inductive formulas.

A common critique of this notion of Predicative is that the extensions by definition, and formulas defining the interpretation will use quantification over all the original “integers”. This is used to argue that it is actually impredicative.

Ed Nelson’s response would surely have been that this critique has no force. As a pure formalist, he is only asserting the *consistency* of predicative theories, as justified by the consistency of Q. From the formalist viewpoint, the *denotations* of formulas in these theories have no meaning.

Examples of predicative principles: [Nelson, PA]

1. Induction on bounded formulas. Bounded formulas may only use quantifiers which are bounded, $\forall x < t$ and $\exists x < t$.

2. Least number principles for bounded formulas.

3. Sequence coding, Gödel numbers for syntactic objects including formulas and proofs. The smash function, $\#$,

$$x \# y = 2^{|x| \cdot |y|}$$

where $|x| \approx \log_2(x)$. Metamathematic concepts including consistency and interpretability and the statements and proofs of Gödel's Incompleteness theorems.

4. All polynomial time computable functions.

Some principles which are **not** predicative include:

1. The totality of exponentiation: *exp*$:= \forall x \exists y (2^x = y)$.

2. Having an inductive initial segment on which superexponentiation $2 \Uparrow x$ is total. Here, $2 \Uparrow 0 = 1$ and $2 \Uparrow (x+1) = 2^{2 \Uparrow x}$.

3. The Gentzen cut elimination theorem.

4. The consistency of the theory Q.

However, w.r.t. **1.**, principles that follow from a *finite* number of uses of exponentiation are predicative. E.g., the tautological consistency of Q and the bounded consistency of Q. Futhermore, Wilkie-Paris showed that any bounded formula which is a consequence of $Q +$ *exp* is predicative.

Thm: [B'06]
(a) The polynomial space (PSPACE) predicates are predicative.
(b) The exponential time (EXPTIME) predicates are predicative.

Thm: Real analysis, up to at least standard theorems on integration can be developed predicatively.

Proof ideas: Jay Hook, in his 1983 Ph.D. thesis does this under the assumption that exponentiation is not total. The assumption can be removed.

In principle, the work of K. Ko and H. Friedman also indicates a way to predicatively develop real analysis since they show integration is in PSPACE (but Ko and Friedman only consider computability, not provability).
See also Fernando and Ferreira on formulations of real analysis in bounded arithmetic.

Synopsis of Predicative Arithmetic

- Begin with a underspecified mass of integers that are closed under successor, addition and multiplication. The latter two closure properties could be replaced by ternary relations; only the assumption of the successor operation is needed. Motivation: models the integers that can be written in unary notation, the integers that can be counted to.
- Using interpretations, especially inductive initial segments, develop a more refined concept of integer. A compellingly effective treatment of much basic mathematics has been done.
- Nelson asked a "compatibility problem" question: If A and B are predicative principles, then must $A \wedge B$ also be predicative? Solovay showed the answer is no, but with a non-appealing example. Open Question: Are there nonetheless useful compatibility results? (If nothing else, $Q + A \wedge B$ is always consistent platonically.)

IV. Inconsistency proof sought

Ed Nelson put a great deal of emphasis into seeking a proof of an actual inconsistency in impredicative theories, including Peano arithmetic (PA) and Primitive Recursive Arithmetic (PRA).

His efforts combined formalized versions of Gödel's second incompleteness theorem, and formalizing Gentzen's cut elimination theorem.

Nelson [HM, p.26]:

> The goal is to produce an explicit superexponentially long recursion and prove that it does not terminate, thereby disproving Church's Thesis from below, demonstrating that finitism is untenable, and proving that Peano arithmetic is inconsistent.
>
> Do you wish me luck?

V. Nelson's Automated Proof Checker

In a 1993 manuscript [NT], and later in [Elements], Ed revisited the development of predicative arithmetic, with a automated proof checker.

He introduces an automated proof checker, *qea*, that is incorporated in his LaTeX files. Theorems are stated and proved in a formal system that is automatically checked by the computer. The technical content of the theorems is similar to [Nelson, PA].

The proof system *qea* is a kind of deduction proof system (similar to a deduction proof system of Fitch, but using very different notations). To illustrate, consider using the axiom

$$\forall x(x + Sy = S(x + y)) \tag{1}$$

to prove

$$x = 0 + x \to Sx = 0 + Sx \tag{2}$$

Written out in full the proof looks like:

$\forall x(x + Sy = S(x + y))$	(1)	Hypothesis (Axiom)
$x = 0 + x \to Sx = 0 + Sx$	(2)	Goal to be proved
$\{$		Assume its negation
$\quad e = 0 + e \wedge Se \neq 0 + Se$	(3)	New variable e for x in $\neg$(2).
$\quad 0 + Se = S(0 + e)$	(4)	Instance of (1).
$\}$		Simple contradiction reached.

By "simple contradiction" is meant a polynomial time test for quasitautological unsatisfiability. More general nesting of assumptions of (negations of) goals is permitted.

Compact representation of the above proof: (x used in place of e).

$$2\{:x \;\; 1;0;x\}$$

From [Nelson, NT]:

Th 158: $x \neq 0 \quad \to \quad x/x = 1.$

158{:x 113;x;x;1;0 16;x 47;x 130;x 3;x·1 134;x }.

Th 159: $x_1 \leq x_2 \quad \to \quad x_1/y \leq x_2/y.$

We have (.1) $y \neq 0$. There is a non-zero u such that $x_2/y + u = x_1/y$, so $x_1 = y \cdot (x_2/y + u) + r_1 = \big((y \cdot (x_2/y) + (y \cdot u)\big) + r_1 = y \cdot (x_2/y) + (y \cdot u + r_1)$. There is a z such that $x_2 = x_1 + z$, so that $x_2 = \big(y \cdot (x_2/y) + (y \cdot u + r_1)\big) + z = (y \cdot (x_2/y) + \big((y \cdot u + r_1) + z\big)$. Consequently, $r_2 = (y \cdot u + r_1) + z = y \cdot u + (r_1 + z)$, so $y \cdot u \leq r_2$ and hence $y \cdot u < y$, which is impossible.

159{:x_1:x_2:y .1{ 156;x_1 156;x_2 16;0 } 113;x_1;y;x_1/y:r_1 113;x_2;y;x_2/y:r_2 98;x_2/y;x_1/y 44;x_2/y;x_1/y:u 10;y;x_2/y;u 9;$y{\cdot}(x_2/y)$;$y{\cdot}u$;r_1 15;x_1;x_2:z 9;$y{\cdot}(x_2/y)$;$y{\cdot}u{+}r_1$;z 9;$y{\cdot}u$;r_1;z 54;$y{\cdot}(x_2/y)$;$(y{\cdot}u{+}r_1){+}z$;r_2 14;$y{\cdot}u$;$r_1{+}z$;r_2 69;$y{\cdot}u$;r_2;y 95;y;u }.

[Nelson, NT, p.89]:

> ... In the not distant future there will be huge data banks of theorems with rapid search procedures to help mathematicians construct proofs of new theorems. ...
>
> But for centuries to come, human mathematicians will not be replaced by computers. We have different search skills. There is a phase transition separating feasible searches from infeasible ones, a phase transition that is roughly described by the distinction between polynomial time algorithms and exponential time algorithms. The latter are in general infeasible; they will remain forever beyond the reach of both people and machines.

VI. Concluding Thoughts

On intuition as a formalist

Nelson [email, 2005]

> ... I admit — proclaim! — the possibility and necessity of intuition about what kinds of formulas can be proved.

On discovery versus invention

Nelson [M&F, p.4]

> Mathematicians no more discover truths than the sculpter discovers the sculpture inside the stone. [...] But, unlike sculpting, our work is tightly constrained, both by the strict requirements of syntax and by the collegial nature of the enterprise. This is how mathematics differs profoundly from art.

On discovery versus invention (bis)

Nelson [S&S, p.2]:

> I have been doing mathematics for 57 years. What is my experience? Do I discover or invent? Am I a James Cook, finding what was already there, or a Thomas Edison, bringing something new into being? [...] Each mathematician will have a different answer to this question, for doing mathematics is personal and persons are different. But my answer is unequivocal: for me, the experience is one of invention.

Nelson [Confessions, p.50]:

> How can I continue to be a mathematician when I have lost my faith in the semantics of mathematics? Why should I want to continue doing mathematics if I no longer believe that numbers and stochastic processes and Hilbert spaces exist? Well, why should a composer want to compose music that is not program music? Mathematics is the last of the arts to become nonrepresentational.

Sources for quotes

[PA] E. Nelson, Predicative Arithmetic, Princeton University Press, 1986.

[M&F] E. Nelson, Mathematics and Faith. Presented at the Jubilee for Men and Women from the World of Learning, The Vatican, 23-24 May 2000.

[HM] E. Nelson, Hilbert's Mistake, 2nd New York Graduate Student Logic Conference, March 2007.

[NT] E. Nelson, untitled, unpublished manuscript, 1993.

[S&S] E. Nelson, Syntax and Semantics, Pontifical Lateran Univ., Rome, 2002. Also, Mendrisio, Switzerland, 2001.

[WS] E. Nelson, Warning Signs of a Possible Collapse of Contemporary Mathematics, presented in San Marino, August 2010.

[Confessions] E. Nelson, Confessions of an Apostate Mathematician, La Sapienza, Rome, 1995.

[A] E. Nelson, Afterword, Diffusion, Quantum Theory and Radically Elementary Mathematics, ed. Faris, 2006.

Many of these are at: http://www.math.princeton.edu/~nelson/.