THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

PROBABILISTIC ALGORITHMIC RANDOMNESS

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Abstract. We introduce martingales defined by probabilistic strategies, in which randomness is used to decide whether to bet. We show that different criteria for the success of computable probabilistic strategies can be used to characterize ML-randomness, computable randomness, and partial computable randomness. Our characterization of ML-randomness partially addresses a critique of Schnorr by formulating ML randomness in terms of a computable process rather than a computably enumerable function.

§1. Introduction. The intuitive notion of what it means for a 0/1 sequence to be algorithmically random is that the sequence should appear completely random to any computable process. This simple idea has led to a rich and complex theory of algorithmic randomness. Most of this theory is based on three important paradigms for defining algorithmic randomness: first, using Martin-Löf tests [10]; second, using algorithmic betting strategies or martingales [15, 17]; and, third, using Kolmogorov information theory and incompressibility [8, 16]. As it turns out, there are a number of natural notions of algorithmically random sequences, including Martin-Löf randomness (1-randomness), partial computable randomness, computable randomness, and Schnorr randomness, among others. A particularly attractive aspect of these, and other, notions of randomness is that they have equivalent definitions in all three paradigms.

Martin-Löf randomness is commonly considered the central notion of algorithmic randomness. There are several reasons for this. First, although there are a number of different natural notions of randomness, Martin-Löf randomness is the strongest of these that does not explicitly use the halting problem or higher levels of the arithmetic hierarchy. Second, Martin-Löf randomness was one of the earliest notions of randomness to be given elegant characterizations in terms of all three paradigms of Martin-Löf tests, martingales, and (prefix-free) Kolmogorov complexity; in addition, Martin-Löf randomness has several other equivalent elegant characterizations, e.g., as Solovay randomness. Third, the theory of Martin-Löf randomness is mathematically elegant and has nice mathematical properties, such as the existence of universal Martin-Löf tests.

On the other hand, already Schnorr [15, 17] critiqued the notion of Martin-Löf randomness as being too strong, based on the fact that the associated martingales

Supported in part by NSF grant DMS-1101228
Supported in part by NSF grant DMS-1060351.

0022-4812/00/0000-0000/$00.00

are only left c.e. functions, not computable functions. The problem is that these left c.e. functions do not correspond in any intuitive way to a computable betting strategy. For this reason, Schnorr proposed two alternate weaker notion of randomness, now known as computable randomness and Schnorr randomness.

In recent years, there has been more widespread recognition that Schnorr's critique casts serious doubt on the status of Martin-Löf randomness as the best model for algorithmic randomness. This has led a number of researchers to seek a characterization of Martin-Löf randomness in terms of more constructive martingales. One example in this direction is the proposal by Hitchcock and Lutz [6] of computable martingale processes; these exactly characterize Martin-Löf randomness [6, 11]. The drawback of computable martingale processes, however, is that they do not correspond to any reasonable algorithmic betting strategy. In another line of work, the open problem of whether Martin-Löf randomness coincides with the Kolmogorov-Loveland (KL) definition of randomness based on non-monotonic computable betting strategies [13, 2, 12, 7] is largely motivated by Schnorr's critique.

The present paper presents a new kind of martingale, one derived from *probabilistic* strategies, that provides a possible answer to Schnorr's critique of Martin-Löf randomness. We present a definition of *probabilistic* betting strategies: these betting strategies can be carried out by a deterministic algorithm with the aid of random coin flips. Our main theorems give exact characterizations of Martin-Löf randomness, partial computable randomness, and computable randomness in terms of these probabilistic betting strategies. We prove that Martin-Löf random sequences are precisely the sequences for which no probabilistic betting strategy has unbounded expected capital, in other words, unbounded expected winnings as the number of bets increases. Computable randomness and partial computable randomness are characterized in terms of having unbounded capital with probability one.

Precise definitions are in the next section, but it is easy to informally describe these probabilistic betting strategies. The probabilistic betting strategy A places a sequence of bets on the bits of a sequence $X \in \{0,1\}^\infty$. Initially, A has capital equal to 1. At each step, the strategy A deterministically computes a probability value $p \in [0,1]$ and a stake value $q \in [0,2]$. At this point, A either bets with probability p, or with probability $1-p$ does not bet at this time. If A bets, it bets on the next unseen bit of X, betting the amount $(q-1)C$ that the bit is zero (equivalently, betting the amount $(1-q)C$ that the next bit is one), where C is the current capital held by A. If the bet is correct, the capital amount is then increased by the bet amount; otherwise, it is decreased by that amount. If A does not bet, the bit of X is not revealed; in the next step, A will again probabilistically decide whether to bet on this bit, possibly changing the probability with which it bets on the next bit and the associated stake.

The probabilistic strategy is defined to be successful against an infinite sequence provided it gains unbounded capital as winnings *in expectation* or, alternately, provided it gains unbounded capital *with probability one*. It is the former definition that gives our new characterization of Martin-Löf randomness.

An advantage of our approach is that, unlike the left c.e. martingales that traditionally correspond to Martin-Löf randomness, our probabilistic betting strategies correspond to algorithms that can actually be carried out. The only non-algorithmic aspect is the use of randomness to decide whether to bet or not at each step. Furthermore, the fact that betting strategies are allowed to use randomness is entirely natural. In practical terms, randomness is feasible to implement, for instance by flipping coins or waiting for atomic decay events. In addition, it seems quite natural that if a sequence is random, then it should also be random relative to most randomly chosen advice strings. An additional motivation is that incorporating randomness into deterministic computation is already widely done in complexity theory to study cryptography and other problems related to the P versus NP problem, see for instance the texts [3, 5]. Our definitions below of probabilistic strategies use a somewhat more restricted version of randomized computation than is common in complexity theory; namely, our probabilistic strategies are allowed to use randomness only when deciding whether or not to place a bet. However, as we show in work in progress, the strength of our probabilistic strategies is unchanged when randomness is allowed at any point instead of just when deciding whether or not to bet.

Section 2 introduces our new notions of Ex-randomness (expected unbounded winnings) and of P1-randomness (unbounded winnings with probability one). The reader may wish to refer to the texts [4, 9, 14] for more background on algorithmic randomness. Section 3 discusses the equivalence of using limsup and lim for the definition of success of probabilistic martingales. Section 4 proves our main equivalences for Martin-Löf randomness. Section 5 proves our new characterizations for computable randomness. Section 6 then establishes a similar characterization for partial computable randomness. Section 7 discusses some counterexamples, showing that certain types of natural definitions for probabilistic betting strategies are too strong to characterize random sequences; these results discuss theorems we initially conjectured to be true, but later discovered to be false. We conclude with some observations and open questions in Section 8.

Our results are summarized in the following figure. The implications and separations are well-known [1, 13, 16]. The equalities involve our new Ex and P1 concepts, and are established in this paper.

$$\boxed{\text{ML-random} = \mathsf{Ex}\text{-random}}$$
$$\Downarrow \; \not\Uparrow$$
$$\boxed{\text{partial computably random} = \mathsf{P1}\text{-random} = \text{locally weak } \mathsf{Ex}\text{-random}}$$
$$\Downarrow \; \not\Uparrow$$
$$\boxed{\text{computably random} = \text{weak } \mathsf{P1}\text{-random} = \text{weak } \mathsf{Ex}\text{-random}}$$

We thank Leszek Kołodziejczyk for suggestions and corrections, and Logan Axon, Denis Hirschfeldt, Bjørn Kjos-Hanssen, Joe Miller, and especially the two anonymous referees for helpful comments.

§2. Preliminaries.

DEFINITION 2.1. Let Γ be a finite alphabet. We denote by Γ^* and Γ^∞ the sets of finite and infinite strings (respectively) over Γ. The empty string is denoted λ. For $\alpha \in \Gamma^* \cup \Gamma^\infty$ and $n \geq 0$, we write $\alpha(n)$ to denote the symbol in position n in α: the first symbol of α is $\alpha(0)$, the second is $\alpha(1)$, etc. For $\beta \in \Gamma^*$, we write

$\beta \sqsubseteq \alpha$ (or $\beta \sqsubset \alpha$) to mean β is a (proper) initial prefix of α. Now suppose $\alpha \in \Gamma^*$. The length of α is denoted $|\alpha|$. We let $|\alpha|_a$ denote the number of occurrences of the symbol a in α. For $\alpha \neq \lambda$, α^- is α minus its last symbol. Also, $[\alpha]$ denotes the set containing the infinite sequences $X \in \Gamma^\infty$ for which $\alpha \sqsubset X$. We write $X \upharpoonright n$ to denote the initial prefix of X of length n. A set $S \subset \Gamma^*$ is *prefix-free* provided there do not exist $\sigma, \sigma' \in S$ with $\sigma \sqsubset \sigma'$.

Recall the well-known definitions associating martingales and algorithmic randomness:

DEFINITION 2.2. A function $d : \{0,1\}^* \to \mathbb{R}^{\geq}$ is a *martingale* if for all $\sigma \in \{0,1\}^*$

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}. \tag{1}$$

It is a *supermartingale* if the equality $=$ in (1) is replaced by the inequality $\geq$. A partial function $d : \ \{0,1\}^* \ \to \ \mathbb{R}^{\geq}$ is a (super)martingale provided, for all $\sigma \in \{0,1\}^*$, if either of $d(\sigma 0)$ or $d(\sigma 1)$ is defined, then equation (1) holds with all of its terms defined. A (super)martingale d *succeeds* on $X \in \{0,1\}^\infty$ if

$$\limsup_{n\to\infty} d(X \upharpoonright n) = \infty. \tag{2}$$

Since a martingale is a real-valued function, it can be classified as computable or computably enumerable using the standard definitions (see, e.g., chapter 5 in [4]). Namely, a martingale d is computable provided there is a rational-valued computable function $f(\sigma, n)$ such that $f(\sigma, n)\downarrow$ iff $d(\sigma)\downarrow$, and $|f(\sigma, n) - d(\sigma)| < 2^{-n}$ for all n and σ. And, a martingale d is computably enumerable provided $\{(\sigma, q) : q \in \mathbb{Q}, q < d(\sigma)\}$ is computably enumerable.

Martingales have been used to define notions of algorithmic randomness. The intuition is that an infinite sequence X is random if no effective betting strategy attains unbounded capital when playing against it. In a fair game, the capital earned by a betting strategy satisfies the martingale property. Therefore, we have the following definitions.

DEFINITION 2.3. The infinite sequence X is called *computably random* if no (total) computable martingale succeeds on it. It is *partial computably random* if no partial computable martingale succeeds on it. And, it is *Martin-Löf random* if no computably enumerable martingale succeeds on it.

PROPOSITION 2.4. *An infinite sequence X is computably random, partial computably random, or ML-random if and only if* $\lim_n d(X \upharpoonright n) \neq \infty$ *for all computable, partial computable, or computably enumerable martingales (respectively).*

PROOF. Theorem 7.1.3 in [4] proves the equivalence in the case of computable and partial computable randomness. For ML-randomness, the proof of Schnorr's theorem (Theorem 6.3.4 in [4]) on the equivalence between the martingale and ML-test characterizations of ML-randomness shows that if X is not ML-random, then this is witnessed by a computably enumerable martingale d such that $\lim_n d(X \upharpoonright n) = \infty$. ⊣

Even though a martingale is a real-valued function, the next proposition states that rational-valued functions suffice for describing (possibly partial) computable randomness. A (partial) computable rational-valued function f is a function for which there is an algorithm which, on input x, halts if and only if $f(x)\downarrow$ and, when it halts, outputs the exact value of $f(x)$.

Proposition 2.5 (Schnorr, as attributed in [4, Prop. 7.1.2]). *An infinite sequence $X \in \{0,1\}^\infty$ is (partial) computably random if and only if no rational-valued (partial) computable martingale succeeds on it.*

Each of these classical martingales corresponds to a betting strategy in which, after seeing σ, the strategy bets that $X(|\sigma|) = 0$ with stake $q(\sigma) = d(\sigma 0)/d(\sigma)$. Our extension to probabilistic strategies A will use both a stake function q_A and a betting probability function p_A. In particular, in addition to the outcome of each bet, we also record decisions of the strategy to "bet" (b) or "wait" (w). The next definition expresses this formally.

Definition 2.6. A *probabilistic strategy* A consists of a pair of rational-valued computable functions $p_A(\pi,\sigma)$ and $q_A(\pi,\sigma)$ such that

$$p_A : \{\mathsf{b},\mathsf{w}\}^* \times \{0,1\}^* \to \mathbb{Q} \cap [0,1], \qquad q_A : \{\mathsf{b},\mathsf{w}\}^* \times \{0,1\}^* \to \mathbb{Q} \cap [0,2].$$

The input $\pi \in \{\mathsf{b},\mathsf{w}\}^*$ is a description of the run of the strategy so far, where b corresponds to a decision to bet and w to wait. The input $\sigma \in \{0,1\}^*$ represents the string of bits that have been bet on so far, an initial prefix of the infinite string being played against. At each step during the run of the strategy, the number of bets placed so far, $|\pi|_{\mathsf{b}}$, should equal the number of bits that have been revealed by the bets, $|\sigma|$. Therefore, we always require that each input pair (π,σ) satisfies $|\pi|_{\mathsf{b}} = |\sigma|$; the values of $p_A(\pi,\sigma)$ and $q_A(\pi,\sigma)$ are irrelevant when this does not hold.

Let $|\sigma| = |\pi|_{\mathsf{b}}$. The intuition is that the value $p_A(\pi,\sigma)$ is the probability that the strategy places a bet during this move:

$$p_A(\pi,\sigma) \;=\; \text{Prob}[A \text{ bets at this step} \mid \pi \text{ describes the bet/wait moves so far in a game played against } X \sqsupset \sigma].^1$$

If A does bet, it will be on the next bit $X(|\sigma|)$ of X. The value $q = q_A(\pi,\sigma)$ is the stake associated with this bet (if it occurs). If $q > 1$, then the strategy is betting that $X(|\sigma|) = 0$; if $q < 1$, the bet is that $X(|\sigma|) = 1$.

The strings $\pi \in \{\mathsf{b},\mathsf{w}\}^*$ form a binary tree called the *computation tree*. The probability that the strategy A follows a particular path through the computation tree depends on the p_A values, and these depend on the so-far revealed bits of the infinite string being played against.

Lemmas 4.6 and 5.4 establish (super)martingale properties for the capital earned by probabilistic strategies while playing on infinite strings.

Definition 2.7. The *cumulative probability* of π relative to σ, $P_A(\pi,\sigma)$, is the probability that the strategy A reaches the node π when running against an infinite string with prefix σ:

$$P_A(\pi,\sigma) \;=\; \text{Prob}[\pi \text{ gives the initial bet/wait moves of } A \mid \sigma \sqsubset X].$$

[1] In expressions involving probabilities, we use "|" to denote conditioning.

The formal definition of P_A proceeds inductively. For the base case, $P_A(\lambda,\lambda)=1$. For non-empty $\pi\in\{\mathsf{b},\mathsf{w}\}^*$,

$$P_A(\pi,\sigma)=\begin{cases}P_A(\pi^-,\sigma^-)\cdot p_A(\pi^-,\sigma^-) & \text{if } \pi=(\pi^-)\mathsf{b}\\ P_A(\pi^-,\sigma)\cdot(1-p_A(\pi^-,\sigma)) & \text{if } \pi=(\pi^-)\mathsf{w}.\end{cases}$$

The *capital* at π relative to σ, $C_A(\pi,\sigma)$, is the amount of capital the strategy has at the node specified by π when playing against an infinite string with prefix σ. We adopt the convention that the initial capital equals 1, so $C_A(\lambda,\lambda)=1$. For non-empty $\pi\in\{\mathsf{b},\mathsf{w}\}^*$, C_A is inductively defined by

$$C_A(\pi,\sigma)=\begin{cases}C_A(\pi^-,\sigma^-)\cdot q_A(\pi^-,\sigma^-) & \text{if } \pi=(\pi^-)\mathsf{b} \text{ and } \sigma=(\sigma^-)0,\\ C_A(\pi^-,\sigma^-)\cdot(2-q_A(\pi^-,\sigma^-)) & \text{if } \pi=(\pi^-)\mathsf{b} \text{ and } \sigma=(\sigma^-)1\\ C_A(\pi^-,\sigma) & \text{if } \pi=(\pi^-)\mathsf{w}.\end{cases}$$

For $X\in\{0,1\}^*$, $p_A^X(\pi)$ abbreviates $p_A^X(\pi,X\upharpoonright|\pi|_{\mathsf{b}})$, and $q_A^X(\pi),P_A^X(\pi),C_A^X(\pi)$ are analogous abbreviations.

LEMMA 2.8. *For A a probabilistic strategy, $\pi\in\{\mathsf{b},\mathsf{w}\}^*$, $\sigma\in\{0,1\}^{|\pi|_{\mathsf{b}}}$,*

$$\sum_{j\in\mathbb{N}}P_A(\pi\mathsf{w}^j,\sigma)p_A(\pi\mathsf{w}^j,\sigma) \;=\; P_A(\pi,\sigma)\Big(1-\prod_{j\in\mathbb{N}}(1-p_A(\pi\mathsf{w}^j,\sigma))\Big). \tag{3}$$

The quantity on the left-hand side of (3) is equal to the probability that, for input $X\in[\sigma]$, the strategy A reaches node π in the computation tree and goes on to place a subsequent bet. The infinite product on the right-hand side of (3) is the probability that the strategy never makes a bet after reaching node π when playing against a string extending σ, conditioned on having reached π.

PROOF. By definition of $P_A(\pi\mathsf{w}^j,\sigma)$, for $j\ge 0$,

$$P_A(\pi\mathsf{w}^j,\sigma) \;=\; P_A(\pi,\sigma)\prod_{k=0}^{j-1}(1-p_A(\pi\mathsf{w}^k,\sigma)).$$

Therefore, it suffices to prove the following holds for $m\ge 0$:

$$\sum_{j=0}^{m}p_A(\pi\mathsf{w}^j,\sigma)\prod_{k=0}^{j-1}(1-p_A(\pi\mathsf{w}^k,\sigma)) \;=\; 1-\prod_{j=0}^{m}(1-p_A(\pi\mathsf{w}^j,\sigma)). \tag{4}$$

This can readily be proved by induction on m. Alternately, and more intuitively, the left-hand side of (4) is the probability that after reaching π, the strategy A bets on its $(j+1)$st attempt (after j wait events) for some $j\le m$; the right-hand side of (4) equals one minus the probability of waiting at least $m+1$ times after reaching π. From this, it is clear that equality holds. $\dashv$

A classical martingale is successful against an infinite string X if it accumulates unbounded capital during the play. In the context of probabilistic computation, there are several ways to define analogous notions.

DEFINITION 2.9. Let A be a probabilistic strategy and let $X\in\{0,1\}^\infty$. Then μ_A^X is the probability distribution on $\{\mathsf{b},\mathsf{w}\}^\infty$ defined on the basic open sets $[\pi]$, $\pi\in\{\mathsf{b},\mathsf{w}\}^*$, by $\mu_A^X([\pi])=P_A^X(\pi)$.

DEFINITION 2.10. Let $\Pi \in \{\mathsf{b},\mathsf{w}\}^\infty$ and $X \in \{0,1\}^\infty$. A probabilistic strategy A *succeeds against* X *along* Π provided

$$\lim_{n\to\infty} C_A^X(\Pi\restriction n) = \infty.$$

Moreover, A *succeeds against* X *with probability one* if

$$\mu_A^X\left(\{\Pi \in \{\mathsf{b},\mathsf{w}\}^\infty : \lim_{n\to\infty} C_A^X(\Pi\restriction n) = \infty\}\right) = 1.$$

In this case, A is a $\mathsf{P1}$-*strategy* for X. The infinite sequence $X \in \{0,1\}^\infty$ is $\mathsf{P1}$-*random* if no probabilistic strategy is a $\mathsf{P1}$-strategy for X.

An alternate definition of success for a probabilistic martingale uses expectation. In particular, we will formalize the intuition of the expected capital of the strategy being unbounded. The definition of expected capital will be given in terms of the number of bets placed; for this, we let $R(n)$ to be the set of computation nodes that can reached immediately after the n-th bet.

DEFINITION 2.11. Let $n \in \mathbb{N}$. Then

$$R(n) \;=\; \{\pi \in \{\mathsf{b},\mathsf{w}\}^* : |\pi|_{\mathsf{b}} = n, \pi \neq \pi^-\mathsf{w}\}.$$

Note that $R(0) = \{\lambda\}$ and that $R(n+1)$ can be expressed in terms of $R(n)$ as

$$R(n+1) \;=\; \bigcup_{\pi\in R(n)} \{\pi\mathsf{w}^j\mathsf{b} : j \in \mathbb{N}\}.$$

DEFINITION 2.12. The *expected capital* after n bets of a probabilistic strategy A over $X \in \{0,1\}^\infty$ is

$$\mathsf{Ex}_A^X(n) = \sum_{\pi\in R(n)} P_A^X(\pi) C_A^X(\pi). \tag{5}$$

The *expected capital* after seeing an initial prefix $\sigma \in \{0,1\}^*$ is $\mathsf{Ex}_A^\sigma = \mathsf{Ex}_A^X(|\sigma|)$ for any X extending σ.

Of course, there may be runs of the strategy A over X that never place n bets. To make sense of $\mathsf{Ex}_A^X(n)$ as an expectation, we define the value of "the capital of A after n bets" to equal zero in the event that A never makes n bets. Then we have,

$$\mathsf{Ex}_A^X(n) \;=\; \text{the expected value for the capital of } A \text{ after } n \text{ bets.}$$

DEFINITION 2.13. A probabilistic strategy A is an Ex-*strategy* for $X \in \{0,1\}^\infty$ if

$$\lim_{n\to\infty} \mathsf{Ex}_A^X(n) = \infty.$$

The infinite sequence $X \in \{0,1\}^\infty$ is Ex-*random* if no probabilistic strategy is an Ex-strategy for X.

We can weaken the above criteria for randomness by only considering probabilistic strategies that don't "get stuck". In general, a probabilistic martingale might reach a state where it never bets on the next bit of X, or more generally has positive probability of never betting on the next bit. This is disallowed by the next definitions.

DEFINITION 2.14. A probabilistic martingale A *always eventually bets with probability one* provided that, for all $\pi \in \{\mathsf{b}, \mathsf{w}\}^*$ and all $\sigma \in \{0,1\}^{|\pi|_\mathsf{b}}$,

$$P_A(\pi, \sigma) \cdot \prod_{i \in \mathbb{N}} (1 - p_A(\pi \mathsf{w}^i, \sigma)) \;=\; 0. \tag{6}$$

The martingale A *eventually bets on X with probability one* provided for all $\pi \in \{\mathsf{b}, \mathsf{w}\}^*$

$$P_A^X(\pi) \cdot \prod_{i \in \mathbb{N}} (1 - p_A^X(\pi \mathsf{w}^i)) \;=\; 0. \tag{7}$$

As in Lemma 2.8, the infinite products in (6) and (7) are equal to the probability that, once node π has been reached, A never places another bet. Thus, these definitions exclude the possibility of A reaching node π with non-zero probability and having zero probability of ever placing another bet. We arrive at weakened versions of probabilistic randomness.

DEFINITION 2.15. A sequence $X \in \{0,1\}^\infty$ is *weak* P1*-random* if no probabilistic martingale which always eventually bets with probability one is a P1 strategy for X.

DEFINITION 2.16. A sequence $X \in \{0,1\}^\infty$ is *weak* Ex*-random* if no probabilistic martingale which always eventually bets with probability one is an Ex-strategy for X.

DEFINITION 2.17. A sequence $X \in \{0,1\}^\infty$ is *locally weak* Ex*-random* if no probabilistic martingale which eventually bets on X with probability one is an Ex-strategy for X.

It is easy to verify that any P1-strategy for X already satisfies the "locally weak"property, so we do not need a definition of "locally weak P1-random".

PROPOSITION 2.18. *Let $X \in \{0,1\}^\infty$.*

(a) *If X is* Ex*-random, then X is locally weak* Ex*-random.*
(b) *If X is locally weak* Ex*-random, then X is weak* Ex*-random.*
(c) *If X is* P1*-random, then X is weak* P1*-random.*
(d) *If X is* Ex*-random, then X is* P1*-random.*
(e) *If X is weak* Ex*-random then X is weak* P1*-random.*
(f) *If X is locally weak* Ex*-random, then X is* P1*-random.*

PROOF. Parts (a)-(c) are immediate from the definitions. Now, suppose A is a P1-strategy for X. The next lemma, Lemma 2.20, shows that A is already an Ex-strategy for X; this suffices to prove parts (d) and (e). Combined with the observation above that any P1-strategy eventually bets with probability one for any X, Lemma 2.20 also gives (f). ⊣

DEFINITION 2.19. The probabilistic strategy A *succeeds with non-zero probability against X* if, for some $T > 0$,

$$\mu_A^X \left(\{\Pi \in \{\mathsf{b}, \mathsf{w}\}^\infty : \lim_{n \to \infty} C_A^X(\Pi \restriction n) = \infty\} \right) = T. \tag{8}$$

LEMMA 2.20. *Suppose that A succeeds against X with non-zero probability. Then A also* Ex*-succeeds against X.*

The proof of Lemma 2.20 formalizes the fact that if a non-zero probability fraction of the runs have capital tending to infinity, then the expected capital (taken over all runs) tends to infinity.

Proof. Suppose (8) holds with $T > 0$. We need to show that $\lim_n \mathsf{Ex}_A^X(n) = \infty$. For $N > 0$ and $s > 0$, define $\mathcal{P}_{s,N}$ as

$$\mathcal{P}_{s,N} \;=\; \{\Pi \in \{\mathsf{b},\mathsf{w}\}^\infty : \forall n \geq N, C_A^X(\Pi\upharpoonright n) > s\}.$$

Fix $s > 0$. Then (8) implies that $\lim_N \mu_A^X(\mathcal{P}_{s,N}) \geq T$. Therefore, there is some N_s such that $\mu_A^X(\mathcal{P}_{s,N_s}) \geq T/2$. Therefore, for all $n \geq N_s$,

$$\mathsf{Ex}_A^X(n) \;=\; \sum_{\pi\in R(n)} P_A^X(\pi)\cdot C_A^X(\pi) \;>\; \mu_A^X(\mathcal{P}_{s,N_s})\cdot s \;\geq\; (T/2)\cdot s.$$

The first inequality follows from the fact that $R(n)$ is a prefix-free cover of $\{\mathsf{b},\mathsf{w}\}^\infty$ and each member of $R(n)$ has length at least $n \geq N_s$. Therefore, some subset of $R(n)$ covers $\mathcal{P}_{s,N_s}$ and hence the sum of $P_A^X(\pi)$ over this set is greater than or equal to $\mu_A^X(\mathcal{P}_{s,N_s})$. For each of the strings, π, in this subset, $C_A^X(\pi) > s$ by definition of $\mathcal{P}_{s,N_s}$.

Taking the limit as $s \to \infty$ gives that $\lim_n \mathsf{Ex}_A^X(n) = \infty$ and proves Proposition 2.18. ⊣

For the next lemma, note that

$$\sum_{\pi\in R(n)} P_A(\pi,\sigma)$$

is equal to the probability that the strategy A places at least n bets when run against $X \in [\sigma]$.

Lemma 2.21. *Let A be a probabilistic strategy and $n \in \mathbb{N}$.*

(a) *Suppose $\sigma \in \{0,1\}^n$. Then*

$$\sum_{\pi\in R(n)} P_A(\pi,\sigma) \;\leq\; 1.$$

(b) *Suppose A always eventually bets with probability one, and let $\sigma \in \{0,1\}^n$. Then,*

$$\sum_{\pi\in R(n)} P_A(\pi,\sigma) \;=\; 1.$$

(c) *Suppose A eventually bets on $X \in \{0,1\}^\infty$ with probability one. Then,*

$$\sum_{\pi\in R(n)} P_A^X(\pi) \;=\; 1.$$

Proof. Part (a) is simply the fact that the probability of betting at least n times is bounded by one. Parts (b) and (c) follow from the definition of A eventually betting with probability one. Formally, induction on n can be used to prove each part. We present the proof of part (b) and then mention the small changes required to adapt it for the other two statements.

The base case of (b) is trivial since $R(0) = \{\lambda\}$ and $P_A(\lambda,\lambda) = 1$. For the induction step,

$$\begin{aligned} \sum_{\pi\in R(n+1)} P_A(\pi,\sigma) &= \sum_{\tau\in R(n)}\sum_{j\in\mathbb{N}} P_A(\tau \mathsf{w}^j \mathsf{b},\sigma) \\ &= \sum_{\tau\in R(n)}\sum_{j\in\mathbb{N}} P_A(\tau \mathsf{w}^j,\sigma^-) p_A(\tau \mathsf{w}^j,\sigma^-) \\ &= \sum_{\tau\in R(n)} P_A(\tau,\sigma^-)\Big(1-\prod_{j\in\mathbb{N}}(1-p_A(\tau\mathsf{w}^j,\sigma^-))\Big) \\ &= \sum_{\tau\in R(n)} P_A(\tau,\sigma^-) \;=\; 1, \end{aligned} \tag{9}$$

where the third equality is Lemma 2.8, the fourth equality follows from (6), and the last equality is the induction hypothesis.

Part (c) is proved in exactly the same way, except that σ is assumed to be an initial prefix of the given infinite string X. For part (a), the last two equalities in (9) are replaced by inequalities. ⊣

§3. Limits and limsups. As we recalled in Proposition 2.4, the classical notions of ML-randomness and (partial) computable randomness can be equivalently defined in terms of either limits or limsups. The notions of P1- and Ex-randomness were defined above in terms of limits; however, as we discuss next, they can be equivalently defined using limsups.

Definition 3.1. A probabilistic strategy A is a *limsup*-P1-*strategy for* X provided that

$$\mu_A^X\left(\{\Pi\in\{\mathsf{b},\mathsf{w}\}^\infty : \limsup_{n\to\infty} C_A^X(\Pi\restriction n)=\infty\}\right)=1.$$

X is *limsup*-P1-*random* if there is no limsup-P1-strategy for X. Similarly, A is a *limsup*-Ex-*strategy for* X provided

$$\limsup_{n\to\infty} \mathsf{Ex}_A^X(n)=\infty.$$

And, X is *limsup*-Ex-*random* if there is no limsup-Ex-strategy for X. The notions of *weak* limsup-P1, and *weak* and *locally weak* limsup-Ex are defined similarly.

Since having $\limsup_n$ equal to infinity is a weaker condition than having the ordinary limit, $\lim_n$, equal to infinity, it is immediate that the "limsup" notions of randomness imply the "lim" notions. In fact, the limsup and lim notions are equivalent. We first state and prove the equivalence of the P1 versions of randomness.

Theorem 3.2. *Let $X\in\{0,1\}^\infty$. Then X is* P1*-random if and only if it is limsup-*P1*-random. Likewise, X is weak* P1*-random if and only if it is weak limsup-*P1*-random.*

Proof. As just remarked, it is sufficient to prove the forward implications. The proof is based on the same "savings trick" that works in the case of classical martingales, see [4, Prop. 6.3.8]. The basic idea is that a probabilistic strategy with an unbounded limsup payoff can be converted into a probabilistic strategy with payoff tending to infinity by occasionally saving (holding back) some of the winnings.

Specifically, given a probabilistic strategy A, we define another probabilistic strategy A' such that $p_{A'}(\pi,\sigma) = p_A(\pi,\sigma)$ for all π,σ (and so $\mu_A^X = \mu_{A'}^X$ for all $X \in \{0,1\}^\infty$), but with a modified stake function that incorporates the savings trick. We must ensure that, for $X \in \{0,1\}^\infty$ and $\Pi \in \{\mathsf{b},\mathsf{w}\}^\infty$, $\limsup_n C_A^X(\Pi{\restriction}n) = \infty$ if and only if $\lim_n C_A^X(\Pi{\restriction}n) = \infty$.

Fix a capital threshold $C_0 > 1$, and a savings increment S_0, where $0 < S_0 < C_0$. The new probabilistic strategy A' acts as follows: A' maintains a "current savings amount", $S(\pi,\sigma)$. Initially, $S(\lambda,\lambda) = 0$. The strategy A' views $S(\pi,\sigma)$ as being permanently saved, and views the remainder of its capital $W(\pi,\sigma) := C_{A'}(\pi,\sigma) - S(\pi,\sigma)$ as its current working capital. In other words, $W(\pi,\sigma)$ is the amount available for wagering at node π when playing against any extension of σ. If the working capital ever rises above the threshold, A' puts more money in the bank. Formally, we set $S(\pi\mathsf{w},\sigma) = S(\pi,\sigma)$ and

$$S(\pi\mathsf{b},\sigma) = \begin{cases} S(\pi,\sigma^-) & \text{if } C_{A'}(\pi\mathsf{b},\sigma) \leq S(\pi,\sigma^-) + C_0 \\ S(\pi,\sigma^-) + \Delta & \text{otherwise,} \end{cases}$$

where Δ is at least S_0 and large enough so that $W(\pi\mathsf{b},\sigma) \leq C_0$. Whenever A' places a bet, it scales the stake value so as to place the same relative wager as A but only on the amount of capital available for wagering. That is,

$$q_{A'}(\pi,\sigma) - 1 = \frac{(q_A(\pi,\sigma)-1)W(\pi,\sigma) + S(\pi,\sigma)}{W(\pi,\sigma) + S(\pi,\sigma)}$$

It is not hard to show that, for every $X \in \{0,1\}^\infty$ and every $\Pi \in \{\mathsf{b},\mathsf{w}\}^\infty$, $\lim_n C_{A'}^X(\Pi{\restriction}n) = \infty$ iff $\limsup_n C_A^X(\Pi{\restriction}n) = \infty$, since if the latter holds, then A''s working capital must exceed its threshold value C_0 infinitely often, and thus its savings amount increases without bound. ⊣

It is not so easy to apply the savings trick to Ex-randomness since "savings" cannot be protected in the same way from events that occur with low probability. Nonetheless, Ex-randomness, locally weak Ex-randomness, and weak Ex-randomness are equivalent to limsup-Ex-randomness, locally weak limsup-Ex-randomness, and weak limsup-Ex-randomness, respectively. We shall prove these equivalences in the next three sections while proving their equivalences to the notions of ML-randomness, partial computable randomness, and computable randomness (respectively).

§4. Theorems and Proofs for Ex-randomness.

THEOREM 4.1. *Suppose $X \in \{0,1\}^\infty$. If X is ML-random, then X is limsup-Ex-random.*

THEOREM 4.2. *Suppose $X \in \{0,1\}^\infty$. If X is Ex-random, then X is ML-random.*

Recalling that limsup-Ex-random trivially implies Ex-random, we get the following equivalences:

Corollary 4.3. *A sequence X is limsup-Ex-random if and only if it is Ex-random, and if and only if it is ML-random.*

A strategy A is called a *universal* Ex-strategy if, for every $X \in \{0,1\}^\infty$, if there is some Ex-strategy for X then A is an Ex-strategy for X. While proving Theorem 4.2, we define a probabilistic strategy which succeeds on exactly the set of infinite sequences covered by a given ML-test (see the definition below of ML-tests). The proof of Theorem 4.2, applied to a universal ML-test, gives the following corollary.

Corollary 4.4. *There is a universal* Ex*-strategy.*

It will be convenient to work with a definition of ML-randomness in terms of ML-tests.

Definition 4.5. A *Martin-Löf test* (ML-test) is a uniformly c.e. sequence of sets U_i, with $\mu(U_i) \leq 2^{-i}$ for all $i \geq 1$ (where μ is Lebesgue measure). Furthermore, without loss of generality, there is an effective algorithm B which enumerates pairs (i,σ) such that $i \geq 1$ and $\sigma \in \{0,1\}^*$ so that

1. Each $U_i = \bigcup\{[\sigma] : (i,\sigma) \text{ is output by } B\}$.
2. For each i, $U_{i+1} \subseteq U_i$.
3. If B outputs both (i,σ) and (i,σ'), then $[\sigma] \cap [\sigma'] = \emptyset$.
4. For each $i > 0$, B outputs infinitely many pairs (i,σ). The σ's of these pairs can be effectively enumerated as $\sigma_{i,0}, \sigma_{i,1}, \sigma_{i,2}, \ldots$.

An infinite sequence $X \in \{0,1\}^\infty$ *fails* the ML-test if $X \in \bigcap_i U_i$. A sequence X is *ML-random* provided it does not fail any ML-test.

We establish two properties of probabilistic strategies before proving Theorem 4.1. The first of these properties is that the average capital accumulated by a probabilistic strategy is a supermartingale.

Lemma 4.6. *If A is a probabilistic strategy and $\sigma \in \{0,1\}^*$ then*

$$\mathsf{Ex}_A^\sigma \;\geq\; \frac{\mathsf{Ex}_A^{\sigma 0} + \mathsf{Ex}_A^{\sigma 1}}{2} \tag{10}$$

Equation (10) is an inequality instead of an equality because of the possibility that A might "get stuck" after betting on the bits of σ and never place a subsequent bet. Compare this to Lemma 5.4.

Proof. For $\sigma \in \{0,1\}^*$ with $|\sigma| = n$, for any $\pi \in \{\mathsf{b},\mathsf{w}\}^*$ with $|\pi|_{\mathsf{b}} = n$, and for any $j \in \mathbb{N}$,

$$P_A(\pi \mathsf{w}^j \mathsf{b}, \sigma 0) = P_A(\pi \mathsf{w}^j, \sigma) p_A(\pi \mathsf{w}^j, \sigma) = P_A(\pi \mathsf{w}^j \mathsf{b}, \sigma 1)$$

and

$$\begin{aligned} C_A(\pi \mathsf{w}^j \mathsf{b}, \sigma 0) + C_A(\pi \mathsf{w}^j \mathsf{b}, \sigma 1) &= C_A(\pi,\sigma)\big(q_A(\pi \mathsf{w}^j, \sigma) + (2 - q_A(\pi \mathsf{w}^j, \sigma))\big) \\ &= 2 C_A(\pi, \sigma). \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\frac{\mathsf{Ex}_A^{\sigma 0} + \mathsf{Ex}_A^{\sigma 1}}{2} \\
&= \frac{1}{2} \sum_{\pi \in R(n)} \sum_{j \in \mathbb{N}} \big(P_A(\pi \mathsf{w}^j \mathsf{b}, \sigma 0) C_A(\pi \mathsf{w}^j \mathsf{b}, \sigma 0) + P_A(\pi \mathsf{w}^j \mathsf{b}, \sigma 1) C_A(\pi \mathsf{w}^j \mathsf{b}, \sigma 1) \big) \\
&= \sum_{\pi \in R(n)} C_A(\pi, \sigma) \sum_{j \in \mathbb{N}} P_A(\pi \mathsf{w}^j, \sigma) p_A(\pi \mathsf{w}^j, \sigma) \\
&= \sum_{\pi \in R(n)} P_A(\pi, \sigma) C_A(\pi, \sigma) \Big(1 - \prod_{j \in \mathbb{N}} (1 - p_A(\pi \mathsf{w}^j, \sigma)) \Big) \\
&\leq \sum_{\pi \in R(n)} P_A(\pi, \sigma) C_A(\pi, \sigma) \;=\; \mathsf{Ex}_A^{\sigma},
\end{aligned}
$$

where the third equality is given by Lemma 2.8. ⊣

LEMMA 4.7. *Let $\sigma_0 \in \{0,1\}^*$, $S \subseteq \{0,1\}^*$. If S is a prefix-free set of extensions of σ_0, and A is a probabilistic strategy, then*

$$\sum_{\sigma \in S} 2^{-|\sigma|} \mathsf{Ex}_A^{\sigma} \;\leq\; 2^{-|\sigma_0|} \mathsf{Ex}_A^{\sigma_0}.$$

PROOF. This lemma is analogous to Kolmogorov's Inequality for classical (super-)martingales and is proved in a similar way [4, Theorem 6.3.3]. To sketch: it is enough to prove the inequality for finite sets S, and this can be done by induction via repeated applications of Lemma 4.6. ⊣

PROOF OF THEOREM 4.1. Suppose A is a limsup-Ex-strategy for $X \in \{0,1\}^\infty$. We will define an ML-test $\{U_i\}_{i \in \mathbb{N}}$ which X fails. Let

$$U_i \;=\; \{Y \in \{0,1\}^\infty : \exists n \left(\mathsf{Ex}_A^Y(n) > 2^i \right)\} \;=\; \bigcup_{\sigma : \mathsf{Ex}_A^{\sigma} > 2^i} [\sigma].$$

These sets are uniformly enumerable since the sum (5) defining Ex_A^{σ} has all its terms computable and non-negative. Hence the values Ex_A^{σ} are uniformly computably approximable from below. To bound $\mu(U_i)$, let S_i be a prefix-free subset of $\{0,1\}^*$ such that $\mathsf{Ex}_A^{\sigma} > 2^i$ for all $\sigma \in S_i$ and such that the union of the cylinders $[\sigma]$ for $\sigma \in S_i$ covers U_i. All strings in S_i extend λ, so by Lemma 4.7

$$\mu(U_i) \;=\; \sum_{\sigma \in S_i} 2^{-|\sigma|} \;<\; 2^{-i} \sum_{\sigma \in S_i} 2^{-|\sigma|} \mathsf{Ex}_A^{\sigma} \;\leq\; 2^{-i} \mathsf{Ex}_A^{\lambda} \;=\; 2^{-i}$$

since $\mathsf{Ex}_A^{\lambda} = 1$.

By assumption on X, $\limsup_n \mathsf{Ex}_A^X(n) = \infty$, and hence for all i there is some n for which $\mathsf{Ex}_A^X(n) > 2^i$. That is, for each i, $X \in U_i$. Therefore, X is not ML-random. ⊣

PROOF OF THEOREM 4.2. Suppose X is not ML-random, as witnessed by some ML-test $\{U_i\}_{i \in \mathbb{N}}$, as enumerated by an algorithm B. The first part of the proof uses B to construct a limsup-Ex-strategy A which is successful against X. At the end of the proof, we will further prove that A can be converted into an Ex-strategy A'.

We think of the strategy A as going through stages. At the beginning of a stage, A has already made bets against the first n bits of X, for some $n \geq 0$, and thus knows the initial prefix $X{\restriction}n$. The strategy A begins running algorithm B to enumerate the $\sigma_{n+1,j}$'s that specify U_{n+1}, for $j = 0, 1, 2, \ldots$. When $\sigma_{n+1,j}$ is enumerated, set $p_{n+1,j} = 2^{n+1}/2^{|\sigma_{n+1,j}|}$. Note that the measure constraint on U_{n+1} implies that $\sum_j p_{n+1,j} \leq 1$. The intuition is that, with probability $p_{n+1,j}$, A will bet all-or-nothing that $X(k) = \sigma_{n+1,j}(k)$ for $n \leq k < |\sigma_{n+1,j}|$. If $X \in [\sigma_{n+1,j}]$ then all of these bets will be correct and the capital accumulated by A will increase by a factor of $2^{|\sigma_{n+1,j}|-n}$. Otherwise, $X \notin [\sigma_{n+1,j}]$ and the capital will drop to zero along this path of the computation.

Formally, we define p_A and q_A inductively. Suppose π is a minimal node for which $p_A(\pi,\sigma)$ and $q_A(\pi,\sigma)$ are not yet defined, and let $n = |\pi|_{\mathsf{b}}$. Then, for each $j \in \mathbb{N}$, define $p_A(\pi\mathsf{w}^j,\sigma)$ so that

$$p_A(\pi\mathsf{w}^j,\sigma)\prod_{i=0}^{j-1}(1 - p_A(\pi\mathsf{w}^i,\sigma)) \; = \; p_{n+1,j}.$$

Note that since $\sum_j p_{n+1,j} \leq 1$, we have $p_A(\pi\mathsf{w}^j,\sigma) \leq 1$. Also, for all $j \geq 0$ and $1 \leq k < |\sigma_{n+1,j}| - n$, define

$$p_A(\pi\mathsf{w}^j\mathsf{b}^k,\sigma) = 1.$$

And, for $j \geq 0$ and $0 \leq k < |\sigma_{n+1,j}| - n$, define

$$q_A(\pi\mathsf{w}^j\mathsf{b}^k,\sigma) \; = \; \begin{cases} 0 & \text{if } \sigma_{n+1,j}(n+k) = 1 \\ 2 & \text{if } \sigma_{n+1,j}(n+k) = 0; \end{cases}$$

Clearly, all p_A and q_A values are computable from the algorithm B for the ML-test, and A is a probabilistic strategy. To prove that A is a limsup-Ex-strategy for X, we analyze the expected capital of A when played against X. We must show that $\limsup_m \mathsf{Ex}_A^X(m) = \infty$.

Since $X \in \bigcap_n U_n$, there is a (unique) sequence $\{\sigma_{n,j'_n}\}_{n\in\mathbb{N}}$ such that $\sigma_{n,j'_n} \sqsubset X$ for each n. This has an infinite subsequence of values $\sigma_{n_1,j_1}, \sigma_{n_2,j_2}, \sigma_{n_3,j_3}, \ldots$ such that $n_1 = 1$ and each $n_{i+1} = |\sigma_{n_i,j_i}| + 1$. We define $\ell_0 = 0$ and $\ell_i = |\sigma_{n_i,j_i}|$, so that $n_{i+1} = \ell_i + 1$. Note that $\ell_i \geq n_i$. Consider the following sequence of nodes π_k in the computation tree:

$$\pi_k \; = \; \mathsf{w}^{j_1}\mathsf{b}^{\ell_1}\mathsf{w}^{j_2}\mathsf{b}^{\ell_2-\ell_1}\cdots\mathsf{w}^{j_k}\mathsf{b}^{\ell_k-\ell_{k-1}}.$$

The nodes π_k are chosen so that, when run against X, every bet made on the computation path to π_k is successful. Since $|\pi_k|_{\mathsf{b}} = \ell_k$, there are ℓ_k many bets placed on this computation path, and since all of them are successful, $C_A^X(\pi_k) = 2^{\ell_k}$. We have

$$\text{(11)} \qquad \mathsf{Ex}_A^X(\ell_k) \; = \sum_{\pi\in R(\ell_k)} P_A^X(\pi)C_A^X(\pi) \; \geq \; P_A^X(\pi_k)C_A^X(\pi_k)$$

$$= 2^{\ell_k}\prod_{i=1}^{k} p_{n_i,j_i} \; = \; 2^{\ell_k}\prod_{i=1}^{k}\frac{2^{n_i}}{2^{\ell_i}} \; = \; 2^{n_1}\prod_{i=1}^{k-1}\frac{2^{n_{i+1}}}{2^{\ell_i}} \; = \; 2^k.$$

The last equality follows from $n_1 = 1$ and $n_{i+1} = \ell_i + 1$. Thus, $\mathsf{Ex}_A^X(\ell_k) \geq 2^k$. Therefore, $\limsup_n \mathsf{Ex}_A(X{\restriction}n) = \infty$.

At this point, we would like to apply a modified savings trick (see the proof of Theorem 3.2) to A to obtain an Ex-strategy A' for X. The computation showing that $\limsup_n \mathsf{Ex}_A(X{\restriction}n) = \infty$ used only the probabilities on a single computation path $\Pi = \mathsf{w}^{j_1}\mathsf{b}^{\ell_1}\mathsf{w}^{j_2}\mathsf{b}^{\ell_2-\ell_1}\mathsf{w}^{j_3}\mathsf{b}^{\ell_3-\ell_2}\cdots$. A naïve application of the savings trick would give a probabilistic strategy such that the path Π is still taken with exactly the same probabilities. The problem with this is that no matter how much capital is "saved", the weighted capital $P_A^X(\pi)C_A^X(\pi)$ can still become arbitrarily small, because the probabilities p_{n_i,j_i} can be arbitrarily small. Thus an alternate savings trick technique is needed: namely, to have the probabilistic strategy choose with a non-zero probability to permanently switch to wagering evenly (with stake value q equal to 1). Once the strategy starts wagering evenly, its weighted capital along this path remains fixed for the rest of the execution.

Specifically, the probabilistic strategy A' is defined to act like A most of the time, but with the following exception: Every time a string σ_{n_k,j_k} has been completely processed, A' next chooses either (a) with probability 1/2, to enter the mode of betting evenly with probability 1 and stake value 1 from that point on; or (b) with probability 1/2, to not bet this step and then continue simulating the strategy A by enumerating the members of U_{ℓ_k+1} where $\ell_k = |\sigma_{n_k,j_k}|$. In particular, if π is the node reached immediately following the processing of σ_{n_k,j_k} then for any $s \geq 0$,

$$P_{A'}(\pi\mathsf{b}^{s+1},\sigma) = P_{A'}(\pi\mathsf{w},\sigma) = \frac{1}{2}P_{A'}(\pi,\sigma) \tag{12}$$

$$C_{A'}(\pi\mathsf{b}^{s+1},\sigma) = C_{A'}(\pi\mathsf{w},\sigma) = C_{A'}(\pi,\sigma). \tag{13}$$

These distinguished computation nodes π will now be π'_k, defined as

$$\pi'_k \;=\; \mathsf{w}^{j_1+1}\mathsf{b}^{\ell_1}\mathsf{w}^{j_2+1}\mathsf{b}^{\ell_2-\ell_1}\cdots\mathsf{w}^{j_k+1}\mathsf{b}^{\ell_k-\ell_{k-1}}.$$

That is, $\pi = \pi'_k$ is the path π_k padded by k many extra w symbols to indicate that A' continued to simulate A after handling each of $\sigma_{n_1,j_1},\ldots,\sigma_{n_k,j_k}$. Following (12) and (13), we can relate the values of $P_{A'}^X$ and $C_{A'}^X$ to P_A^X and C_A^X: for $s \geq 0$

$$P_{A'}^X(\pi'_k\mathsf{b}^{s+1}) = P_{A'}^X(\pi'_k\mathsf{w}) = 2^{-(k+1)}\cdot P_A^X(\pi_k)$$

$$C_{A'}^X(\pi'_k\mathsf{b}^{s+1}) = C_{A'}^X(\pi'_k\mathsf{w}) = C_A^X(\pi_k).$$

By the above and by the string of equalities in (11),

$$P_{A'}^X(\pi'_k\mathsf{b}^{s+1})C_{A'}^X(\pi'_k\mathsf{b}^{s+1}) = \left(2^{-(k+1)}\cdot P_A^X(\pi_k)\right)C_A^X(\pi_k) = 2^{-(k+1)}2^k = \frac{1}{2}.$$

Therefore, for each n,

$$\begin{aligned}\mathsf{Ex}_{A'}^X(n) &= \sum_{\pi\in R(n)} P_{A'}^X(\pi)C_{A'}^X(\pi)\\ &\geq \sum_{k:\ \ell_k<n} P_{A'}^X(\pi'_k\mathsf{b}^{n-\ell_k})C_{A'}^X(\pi'_k\mathsf{b}^{n-\ell_k}) \;=\; \sum_{k:\,\ell_k<n}\tfrac{1}{2}.\end{aligned}$$

Since the sequence of ℓ_k values is infinite, this sum tends to ∞ as $n\to\infty$. It follows that $\lim_n \mathsf{Ex}_{A'}^X(n) = \infty$ as desired, and X is not Ex-random. $\dashv$

It is interesting to note that both the limsup and the lim versions of the probabilistic martingale described above are oblivious to a certain extent. Namely, in defining A (and A'), the probability $p_{n+1,j}$ is set independently of whether or not $X{\upharpoonright}n = \sigma_{n+1,j}{\upharpoonright}n$. Of course, if they are not equal, it would make more sense to set $p_{n+1,j} = 0$. However, it does not appear that taking this into account would lead to any improvement in the analysis in the proof of Theorem 4.2.

§5. Theorems and Proofs for weak P1-randomness.

THEOREM 5.1. *Suppose $X \in \{0,1\}^\infty$. If X is weak* P1*-random, then X is computably random.*

THEOREM 5.2. *Suppose $X \in \{0,1\}^\infty$. If X is computably random, then X is weak limsup-*Ex*-random.*

As an immediate corollary of Proposition 2.18(e), Theorems 3.2, 5.1, and 5.2, and the fact that weak limsup-Ex-randomness trivially implies weak Ex-randomness, we obtain the following set of equivalences.

COROLLARY 5.3. *The following notions are equivalent: weak* P1*-random, weak limsup-*P1*-random, weak* Ex*-random, weak limsup-*Ex*-random, and computably random.*

PROOF OF THEOREM 5.1. Suppose X is not computably random, and let d be a total computable rational-valued martingale with $\lim_n d(X{\upharpoonright}n) = \infty$. The martingale d immediately gives a probabilistic strategy; namely, for each $\pi \in \{\mathsf{b},\mathsf{w}\}^*$ and $\sigma \in \{0,1\}^{|\pi|_\mathsf{b}}$, $p_d(\pi,\sigma) = 1$ and for each $n \in \mathbb{N}$,

$$q_d(\mathsf{b}^n,\sigma) = \frac{d(\sigma 0)}{d(\sigma)}$$

In particular, there is exactly one infinite path through $\{\mathsf{b},\mathsf{w}\}^*$ with non-zero probability, and along this path, the capital accumulated by the probabilistic strategy is exactly equal to the martingale d. Hence this is a P1-strategy for X. Moreover, it always eventually bets with probability one since d is total and all bets are made with probability one. It follows that X is not weak P1-random. ⊣

PROOF OF THEOREM 5.2. Suppose X is not weak limsup-Ex-random, and let A be a limsup-Ex-strategy for X which always eventually bets with probability one. Define $d : \{0,1\}^* \to \mathbb{R}^{\geq}$ to be $d(\sigma) = \mathsf{Ex}^\sigma_A$.

LEMMA 5.4. *If A always eventually bets with probability one, the function d satisfies the martingale property (1).*

PROOF. By Lemma 4.6, $d(\sigma)$ is a supermartingale. However, since A always eventually bets with probability one, (6) gives that for $\pi \in R(|\sigma|)$,

$$P_A(\pi,\sigma)C_A(\pi,\sigma)\Big(1 - \prod_{j\in\mathbb{N}}(1 - p_A(\pi\mathsf{w}^j,\sigma))\Big) \;=\; P_A(\pi,\sigma)C_A(\pi,\sigma).$$

Hence, the inequality in the proof of Lemma 4.6 can be replaced by equality in this case, and $d(\sigma)$ is a martingale. ⊣

Since A is a (weak) limsup-Ex-strategy, $\limsup_n d(X\upharpoonright n) = \infty$. Thus, Theorem 5.2 will be proved if we can show that d is computable. Since d is a martingale by Lemma 5.4 and since $d(\lambda) = 1$,

$$\sum_{\tau \in \{0,1\}^n} d(\tau) = 2^n \tag{14}$$

for all n. Define the approximation to d at level $M > 0$ to be

$$d(\tau, M) \;=\; \sum_{\pi \in R(|\tau|) : |\pi|_{\mathsf{w}} < M} P_A(\pi, \tau) C_A(\pi, \tau).$$

This is a finite sum of computable terms and approaches $d(\tau)$ from below. It suffices to describe an algorithm which, given $\sigma \in \{0,1\}^*$ and $\epsilon > 0$, approximates $d(\sigma)$ to within ϵ of the true value. To do so, compute

$$\sum_{\tau \in \{0,1\}^{|\sigma|}} d(\tau, M)$$

for increasingly large values of M, until a value for M is found satisfying that this sum is greater than $2^{|\sigma|} - \epsilon$. By (14), this value of M puts the value of $d(\sigma, M)$ within ϵ of $d(\sigma)$. This shows d is computable, and proves Theorem 5.2. ⊣

§6. Theorems and Proofs for $\mathsf{P1}$-randomness.

THEOREM 6.1. *Suppose $X \in \{0,1\}^\infty$. If X is* $\mathsf{P1}$*-random, then X is partial computably random.*

THEOREM 6.2. *Suppose $X \in \{0,1\}^\infty$. If X is partial computably random, then X is locally weak limsup-*Ex*-random.*

As an immediate corollary of Proposition 2.18(f), Theorems 3.2, 6.1, and 6.2, and the fact that locally weak limsup-Ex-randomness trivially implies locally weak Ex-randomness, we obtain the following set of equivalences.

COROLLARY 6.3. *The following notions are equivalent:* $\mathsf{P1}$*-random, limsup-*$\mathsf{P1}$*-random, locally weak* Ex*-random, locally weak limsup-*Ex*-random, and partial computably random.*

PROOF OF THEOREM 6.1. Suppose d is a rational-valued partial computable martingale which succeeds on X. We will define a probabilistic strategy A that eventually bets on X with probability one and is a $\mathsf{P1}$-strategy for X. The idea is that A waits to bet on σ until it has seen that both $d(\sigma)\downarrow$ and $d(\sigma 0)\downarrow$ and, at that point, bets the appropriate stake with probability one. Formally, define for $\pi \in \{\mathsf{b}, \mathsf{w}\}^*$ and $\sigma \in \{0,1\}^{|\pi|_{\mathsf{b}}}$,

$$p_A(\pi, \sigma) \;=\; \begin{cases} 1 & \text{if it takes } |\pi|_{\mathsf{w}} \text{ steps for both } d(\sigma), d(\sigma 0) \text{ to have converged,} \\ 0 & \text{otherwise.} \end{cases}$$

And,

$$q_A(\pi, \sigma) \;=\; \begin{cases} \frac{d(\sigma 0)}{d(\sigma)} & \text{if } p_A(\pi, \sigma) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then, when run against X, all but one of the infinite paths through the computation tree have zero probability. Moreover, since $d(X\restriction n)\downarrow$ for all n, there is a (unique) path with infinitely many bets that is taken with probability one during the run of the strategy on X. On this probability one path, A behaves exactly as d would on X. Thus, A is a weak $\mathsf{P1}$-strategy for X. $\dashv$

Proof of Theorem 6.2. Let $X \in \{0,1\}^\infty$ and suppose A is a limsup-Ex-strategy for X that eventually bets on X with probability one. We wish to define a rational-valued partial computable martingale that succeeds on X.

We will actually define a rational-valued partial computable *supermartingale* d that succeeds on X. This will suffice since it is possible to use d to define a rational-valued partial computable martingale d_0 such that for all σ

$$d_0(\sigma) \geq d(\sigma).$$

In particular, if $\limsup_n d(X\restriction n) = \infty$ then also $\limsup_n d_0(X\restriction n) = \infty$. The construction of d_0 from d is well-known and can be found in [14, 7.1.6]: namely,

$$d_0(\sigma) \;=\; d(\sigma) + \sum_{\sigma' \sqsubset \sigma} \left(d(\sigma') - \frac{d(\sigma' 0) + d(\sigma' 1)}{2} \right).$$

The intuition is that the supermartingale $d(\sigma)$ outputs an approximation to Ex_A^σ when there is evidence that A eventually bets after seeing σ with sufficiently high probability. We will prove that $d(X\restriction n)$ is defined for all n and, more generally, that $d(\sigma)$ satisfies

$$|d(\sigma) - \mathsf{Ex}_A^\sigma| \;\leq\; 1 \tag{15}$$

whenever $d(\sigma)$ is defined. In particular, since $\limsup_n \mathsf{Ex}_A^X(n) = \infty$, it must be that $\limsup_n d(X\restriction n) = \infty$.

We first define a partial computable function $M : \{0,1\}^* \to \mathbb{N}$ by

$$M(\sigma) \;=\; \text{the least } M \text{ s.t.} \sum_{\pi \in R(|\sigma|) : |\pi|_{\mathsf{w}} < M} P_A(\pi, \sigma) \geq 1 - 2^{-2|\sigma|}.$$

That is, $M(\sigma)$ is the threshold "w-distance" required to guarantee that, with high probability, every bit of σ is bet on. The intuition is that $M(\sigma)$ gives the number of terms needed to get a good approximation to the value of Ex_A^σ. Lemma 2.21(a) implies that

$$\sum_{\pi \in R(|\sigma|) : |\pi|_{\mathsf{w}} \geq M(\sigma)} P_A(\pi, \sigma) \;\leq\; 2^{-2|\sigma|} \tag{16}$$

provided $M(\sigma)$ is defined. Since A eventually bets on X with probability one, Lemma 2.21(b) implies that $M(\sigma)$ is defined for all $\sigma \sqsubset X$.

We use another auxiliary computable function, $f : \mathbb{N} \to \mathbb{Q}$, given by $f(n) = 2^{1-n} - 1$. This function has a useful inductive definition that we will exploit: $f(0) = 1$ and $f(n+1) = f(n) - 2^{-n}$. We have $-1 < f(n) \leq 1$ for all n, and $f(n) \leq 0$ for all $n \geq 1$.

Define $d : \{0,1\}^* \to \mathbb{Q}$ to be the partial computable function

$$d(\sigma) \;=\; f(|\sigma|) + \sum_{\pi \in R(|\sigma|) : |\pi|_{\mathsf{w}} < M(\sigma)} P_A(\pi, \sigma) C_A(\pi, \sigma). \tag{17}$$

Note that $d(\sigma)$ is undefined if and only if $M(\sigma)$ is undefined. In particular, $d(\sigma)\downarrow$ for all $\sigma \sqsubset X$. By the definition of Ex_A^σ,

$$d(\sigma) \ = \ \mathsf{Ex}_A^\sigma + f(|\sigma|) - \sum_{\pi \in R(|\sigma|):|\pi|_{\mathsf{w}} \geq M(\sigma)} P_A(\pi,\sigma)C_A(\pi,\sigma). \tag{18}$$

$C_A(\pi,\sigma)$ is the capital accumulated after betting $|\pi|_{\mathsf{b}}$ many times on σ, and each bet can at most double the capital. Therefore, $C_A(\pi,\sigma) \leq 2^{|\pi|_{\mathsf{b}}} = 2^{|\sigma|}$ for $\pi \in R(|\sigma|)$. This fact and (16) imply that

$$\sum_{\pi \in R(|\sigma|):|\pi|_{\mathsf{w}} \geq M(\sigma)} P_A(\pi,\sigma)C_A(\pi,\sigma) \ \leq \ 2^{-|\sigma|}. \tag{19}$$

Combining (18), (19), and the definition of f, we get

$$\mathsf{Ex}_A^\sigma - (1 - 2^{-|\sigma|}) \ \leq \ d(\sigma) \ \leq \ \mathsf{Ex}_A^\sigma - (1 - 2^{1-|\sigma|})$$

whenever $d(\sigma)$ is defined. It follows that (15) holds.

We have shown that d is partial computable and that, for all $\sigma \sqsubset X$, $d(\sigma)$ is defined and approximates Ex_A^σ with bounded error. It remains to prove that d is a supermartingale.

It is a simple observation that $M(\sigma 0) = M(\sigma 1)$ since $P_A(\pi,\sigma 0) = P_A(\pi,\sigma 1)$ holds whenever $\pi \in R(|\sigma|+1)$. In addition, if $M(\sigma 0)$ is defined then $M(\sigma)$ is defined and $M(\sigma) \leq M(\sigma 0)$. This is because the w-distance $M(\sigma 0)$ which suffices to guarantee that all bits of $\sigma 0$ are bet on with high probability certainly suffices to guarantee that all bits of σ are bet on with at least the same probability. Therefore, if either $d(\sigma 0)\downarrow$ or $d(\sigma 1)\downarrow$, then all three of $d(\sigma)\downarrow$, $d(\sigma 0)\downarrow$, and $d(\sigma 1)\downarrow$.

Finally, we prove that the supermartingale property holds for d. We have

$$\begin{aligned}
&d(\sigma 0) + d(\sigma 1) \\
&= \mathsf{Ex}_A^{\sigma 0} + \mathsf{Ex}_A^{\sigma 1} + 2f(|\sigma|+1) \\
&\qquad - \sum_{\pi \in R(|\sigma|+1):|\pi|_{\mathsf{w}} \geq M(\sigma 0)} P_A(\pi,\sigma 0)(C_A(\pi,\sigma 0) + C_A(\pi,\sigma 1)) \\
&\leq \mathsf{Ex}_A^{\sigma 0} + \mathsf{Ex}_A^{\sigma 1} + 2f(|\sigma|+1) \\
&\leq 2(\mathsf{Ex}_A^{\sigma} + f(|\sigma|) - 2^{-|\sigma|}) \\
&\leq 2 \cdot d(\sigma)
\end{aligned}$$

where the second inequality uses the supermartingale property for Ex_A^σ from Lemma 4.6 and the definitions of f and C_A, and the third inequality follows from (18) and (19). This establishes the supermartingale property $d(\sigma 0) + d(\sigma 1) \leq 2d(\sigma)$ for all σ. ⊣

The proof of Theorem 6.1 yields yet another characterization of partial computable randomness. Consider probabilistic strategies where, at each stage, the probability of betting is either zero or one. That is, $p_A(\pi,\sigma) \in \{0,1\}$ for all π,σ. A probabilistic strategy with this property that succeeds on an infinite sequence X is called a w-*strategy* for X. We say that X is w-*random* if there is no w-strategy for X. Intuitively, w-strategies can be seen as interpolating between classical (non probabilistic) strategies and probabilistic strategies. Nonetheless, the following equivalence holds.

Corollary 6.4. *The following notions are equivalent:* $\mathsf{P1}$*-random and* w*-random.*

§7. Counterexample to an alternate definition. The definition of Ex-randomness is based on unbounded expected success of a probabilistic strategy A with respect to snapshots of the computation of A at finite times. Specifically, the expected capital value is computed over computation paths $\pi \in R(n)$. Before defining Ex-randomness in this way, we considered a more general alternate definition: namely, we considered studying nested sequences of arbitrary finite portions of the computation tree and defining Ex-randomness in terms of the expected capital at the leaves of these partial computation trees. However, this turned out to be too powerful a notion, as it excludes all but measure zero many strings from being random, contradicting our intuition that "typical" sequences are random.

Since this notion of randomness seemed very natural to us, we feel it is interesting to present the counterexample which convinced us that this attempt at a more general notion of randomness had failed. The counterexample is interesting also for the reason that it can be adapted to rule out other possible definitions of randomness.

In the definitions below, the definition of a probabilistic strategy A is unchanged; the only difference is the definition of the expected capital of the strategy.

Definition 7.1. A *partial computation tree* is a finite set $f \subseteq \{\mathsf{b},\mathsf{w}\}^*$ which is downward closed and whose maximal elements cover all computation paths. That is, if $\pi \in f$ then all $\pi' \sqsubset \pi$ are in f, and $\pi\mathsf{b} \in f \Leftrightarrow \pi\mathsf{w} \in f$. Thus, f is a binary tree. A maximal node $\pi \in f$ is called a *leaf node*.

Definition 7.2. A sequence $\{f_i\}_{i\in\mathbb{N}}$ of partial computation trees is *nested* if $f_i \subseteq f_j$ for all $i \le j$.

Definition 7.3. Let f be a partial computation tree. The *expected capital earned by strategy A playing on $X \in \{0,1\}^\infty$ up to f* is given by

$$\mathsf{Ex}_A^X(f) \;=\; \sum_{\pi \text{ a leaf of } f} P_A^X(\pi) C_A^X(\pi).$$

We say that $X \in \{0,1\}^\infty$ is *$\mathtt{I}$-random* if there is no probabilistic strategy A and computable sequence of nested partial computation trees $\{f_i\}_{i\in\mathbb{N}}$ such that $\limsup_n \mathsf{Ex}_A^X(f_n) \;=\; \infty$.

Note that the sets $\{R(n)\}_{n\in\mathbb{N}}$ in the definition of Ex-random play the roles of the sets $\{f_i\}_{i\in\mathbb{N}}$ in definition of $\mathtt{I}$-random. However, since each $R(n)$ is infinite, the sequence $\{R(n)\}_{n\in\mathbb{N}}$ cannot witness the success of a strategy in the $\mathtt{I}$-randomness setting. In developing the theory of probabilistic strategies, we initially considered using $\mathtt{I}$-random in place of Ex-random. However, we were quite surprised to discover that nearly no A is $\mathtt{I}$-random. (The "$\mathtt{I}$" stands for "(nearly) impossibly".) That is, for almost all X, there is a probabilistic strategy that succeeds on X in the sense of $\mathtt{I}$-randomness. In fact, and even more surprisingly, there is a single choice for A and $\{f_i\}_{i\in\mathbb{N}}$ that works for almost all X:

THEOREM 7.4. *There is a probabilistic strategy A and a computable sequence of nested partial computation trees $\mathcal{F} = \{f_i\}_{i\in\mathbb{N}}$ such that*

$$\mu\{X : \limsup_{n\to\infty} \mathsf{Ex}_A^X(f_n) = \infty\} \;=\; 1.$$

The measure, μ, is Lebesgue measure.

The probabilistic strategy A is very simple; the complexity in the proof lies in the choice of partial computation trees. The algorithm for A does the following, starting with Step α:

Step α: With probability 1/2, bet all-or-nothing (stake $q = 2$) that the next bit is 0, and return to Step α. Otherwise place no bet (that is, wait) and go to Step β.

Step β: With probability 1, bet all-or-nothing (stake $q = 0$) that the next bit is 1. Then go to Step α.

The strategy A is not biased towards any particular sequence $X \in \{0,1\}^\infty$. Indeed, for each bit, A places two bets with net probabilities 1/2 each: first that the bit equals 0 and then that the bit equals 1. It is the partial computation trees f_i that will bias the expectation towards particular sequences X.

LEMMA 7.5. *Let $K \geq \frac{3}{2}$ and $\epsilon > 0$. There is a finite nested sequence $\{f_i\}_{i\leq L}$ such that*

$$\mu\{X : \max_i \mathsf{Ex}_A^X(f_i) \geq K\} \;\geq\; 1-\epsilon.$$

And, for all X, there is at least one leaf node π of f_L such that $P_A^X(\pi)C_A^X(\pi) > 0$.

The proof of Lemma 7.5 will show that L and the f_i's are uniformly constructible from K and ϵ. Before proving the lemma, we sketch how it implies Theorem 7.4.

PROOF SKETCH OF THEOREM 7.4 FROM LEMMA 7.5. Choose an unbounded increasing sequence of values K_j, say $K_j = j+1$. Let $\epsilon_j = 2^{-j}$, so $\lim_j \epsilon_j = 0$. Initially pick the finite sequence $\mathcal{F}_1$ of partial computation trees $\{f_i\}_{i\leq L_1}$ as given by Lemma 7.5 with $K = K_1$ and $\epsilon = \epsilon_1$.

Suppose $\mathcal{F}_j$ has already been chosen as a nested sequence of L_j+1 many partial computation trees; we extend $\mathcal{F}_j$ to a nested sequence $\mathcal{F}_{j+1}$ of length $L_{j+1}+1$. Let f_{L_j} be the final partial computation tree in $\mathcal{F}_j$. Let $n = \max\{|\pi|_{\mathsf{b}} : \pi \in f_{L_j}\}$. Then the behavior of A on any X up to f_{L_j} is determined by the first n bits of X. Lemma 7.5 guarantees that for each $\sigma \in \{0,1\}^n$ there is at least one leaf node $\pi \in f_{L_j}$ with $P_A(\pi,\sigma)C_A(\pi,\sigma) > 0$; call the least of these nonzero $P_A(\pi,\sigma)C_A(\pi,\sigma)$ values w_σ. Then, define $w = \min\{w_\sigma : \sigma \in \{0,1\}^n\}$. Note that $w > 0$ and is computable from f_{L_j} and A. Let $K = K_{j+1}/w$ and $\epsilon = \epsilon_{j+1}$, and choose $\{f'_{i'}\}_{i'\leq L'}$ as given by Lemma 7.5 for these parameters. Define $f_{L_j}\circ f_{i'}$ to be the result of attaching a copy of $f_{i'}$ to each leaf of f_{L_j}; namely, to be the partial computation tree containing the strings $\pi \in f_{L_j}$ plus the strings $\pi\pi'$ such that π is a leaf node in f_{L_j} and π' is in $f_{i'}$. Finally define $\mathcal{F}_{j+1}$ to be the sequence $\mathcal{F}_j$ extended with the partial computation trees $f_{L_j}\circ f_{i'}$ for $i' \leq L'$.

It is not hard to show that the X's that have $\mathsf{Ex}_A^X(f) \geq K_{j+1}$ for some $f \in \mathcal{F}_{j+1}$ form a set of measure $\geq 1-\epsilon_{j+1}$. Now, form the infinite sequence $\mathcal{F}$ by taking the union of the sequences $\mathcal{F}_j$. ⊣

Proof sketch for Lemma 7.5. The proof is by induction. The base case is $K = 3/2$. After that, we argue that if the lemma holds for K, then it holds also for $K + 1/2$.

Let $K = 3/2$ and $\epsilon = 2^{-j}$. Define the strings $\pi_i = (\mathsf{wb})^i$; namely, π_i represents the situation where, for i times in a row, the strategy A does not bet (w) in Step α and does bet (b) in Step β. Define $f_i = \{\pi_k, \pi_k^- \mathsf{w}, \pi_k \mathsf{b}, \pi_k \mathsf{w} : k \le i\}$. Clearly, each f_i is a partial computation tree and the sequence $\{f_i\}_{i \in \mathbb{N}}$ is nested and computable. Suppose that $1^k 0 \sqsubset X$. It is straightforward to calculate that $P_A^X(\pi_k \mathsf{b}) C_A^X(\pi_k \mathsf{b}) = 2^{-(k+1)} 2^{k+1} = 1$, and $P_A^X(\pi_k \mathsf{w}) C_A^X(\pi_k \mathsf{w}) = 2^{-(k+1)} 2^k = 1/2$. Therefore, $\mathsf{Ex}_A^X(f_k) \ge 3/2$. (In fact, equality holds, as all other leaves have capital equal to zero for X.) Letting $\mathcal{F} = \{f_i\}_{i \le j}$, this suffices to prove the $K = 3/2$ case of the lemma since only a fraction $\epsilon = 2^{-j}$ of X's start with 1^j.

Now suppose we have already constructed a sequence $\mathcal{F}' = \{f'_{i'}\}_{i' \le L'}$ which satisfies the lemma for K with $\epsilon = 2^{-(j+1)}$. We will construct a sequence $\mathcal{F}''$ that works for $K + 1/2$ and $\epsilon = 2^{-j}$. The idea is to start with the $\mathcal{F} = \{f_i\}_{i \le L}$ just constructed for the $K = 3/2$ and $\epsilon = 2^{-(j+1)}$ case. We interleave the construction of members of $\mathcal{F}$ with copies of $\mathcal{F}'$ added at the leaf node $\pi_i \mathsf{b}$ of each $f_i \in \mathcal{F}$. For this define $f''_0 = f_0$ and

$$f''_{i,i'} \;=\; f''_i \cup \pi_i \mathsf{b} f'_{i'} \qquad \text{and} \qquad f''_{i+1} \;=\; f''_{i,L'} \cup f_{i+1}.$$

where $\pi_i \mathsf{b} f'_{i'} = \{\pi_i \mathsf{b} \tau : \tau \in f'_{i'}\}$. This forms a nested sequence $\mathcal{F}''$ of partial computation trees, $f''_0, f''_{0,0}, \ldots, f''_{0,L'}, f''_1, f''_{1,0}, \ldots, f''_{1,L'}, f''_2, \ldots, f''_{L,L'}$. We leave it to the reader to verify that $\mathcal{F}''$ satisfies the desired conditions of Lemma 7.5. ⊣

§8. Conclusions and Open Questions. We conclude with a few open problems. First, we ask for characterizations of other notions of randomness in terms of probabilistic strategies. In particular, what natural conditions on the class of strategies are equivalent to Schnorr randomness? Algorithmic randomness has also been studied relative to weaker models of computation (primitive recursive functions, polynomial-time computation, etc.). Do probabilistic strategies shed light on randomness in this setting as well?

Second, we note that the definition of unbounded expected success requires some subtlety. In Section 4 we showed that defining this notion via Ex_A^X characterizes ML-randomness, whereas Section 7 showed that another definition of expected value gives a trivial notion of randomness. One can redefine $\mathtt{I}$-randomness to use $\lim_n$ instead of $\limsup_n$. Does this revised version notion of $\mathtt{I}$-randomness make sense? Does the set of $\mathtt{I}$-random sequences now have measure one?

Finally, what happens when the definition of probabilistic strategies is extended to non-monotonic Kolmogorov-Loveland randomness? Consider non-monotonic probabilistic strategies which are similar to the probabilistic strategies defined here but also have an integer-valued function $n_A(\pi, \sigma)$ which specifies which bit to bet on. The string σ now codes the values of the bits $X(n)$ that have already been bet upon, and the index $n_A(\pi, \sigma)$ must not specify a bit that has already been bet on. The non-monotonic probabilistic strategy A bets with probability $p_A(\pi, \sigma)$ on the bit $X(n_X(\pi, \sigma))$ using stake value $q_A(\pi, \sigma)$. Is there a coherent definition of Ex-randomness that applies to non-monotonic probabilistic strategies; for instance, one that has a set of measure one as its random sequences? Unfortunately, it is possible to adapt Theorem 7.4 to show that the obvious way of defining limsup-Ex-randomness for non-monotonic probabilistic martingales does not work. In another direction, it is straightforward to extend the notions of P1-randomness to non-monotonic probabilistic strategies. How do these various definitions of P1-randomness for non-monotonic strategies compare to each? How do they compare to Martin-Löf randomness and Kolmogorov-Loveland randomness?

REFERENCES

[1] Klaus Ambos-Spies, *Algorithmic randomness revisited*, ***Language, logic, and formalization of knowledge, Coimbra lecture and proceedings of a symposition held in Siena in September 1997***, Bibliotheca, 1998, pp. 33–52.

[2] Klaus Ambos-Spies and Antonín Kučera, *Randomness in computability theory*, ***Computability theory and its applications, proceedings an a 1999 ams-ims-siam joint summer research conference***, American Mathematics Society, 2000, pp. 1–14.

[3] Sanjeev Arora and Boaz Barak, ***Computational complexity: A modern approach***, Cambridge University Press, 2009.

[4] Rodney G. Downey and Denis Hirschfeldt, ***Algorithmic randomness and complexity***, Springer, 2010.

[5] Oded Goldreich, ***Computational complexity: A conceptual perspective***, Cambridge University Press, 2008.

[6] John M. Hitchcock and Jack H. Lutz, *Why computable complexity requires stricter martingales*, ***Theory of Computing Systems***, vol. 39 (2006), pp. 277–296.

[7] Bart Kastermans and Steffen Lempp, *Comparing notions of randomness*, ***Theoretical Computer Science***, vol. 411 (2010), no. 3, pp. 602–616.

[8] Leonid Levin, *On the notion of a random sequence*, ***Soviet Mathematics Doklady***, vol. 14 (1973), pp. 1413–1416.

[9] Ming Li and Paul Vitányi, ***An introduction to Kolmogorov complexity and its applications***, third ed., Springer, 2008.

[10] Per Martin-Löf, *The definition of random sequences*, ***Information and Control***, vol. 9 (1966), no. 6, pp. 602–619.

[11] Wolfgang Merkle, Nenad Mihailović, and Theodore A. Slaman, *Some results on effective randomness*, ***Theory of Computing Systems***, vol. 39 (2006), pp. 707–721.

[12] Wolfgang Merkle, Joseph S. Miller, André Nies, Jan Reimann, and Frank Stephan, *Kolmogorov-Loveland randomness and stochasticity*, ***Annals of Pure and Applied Logic***, vol. 103 (2006), pp. 183–210.

[13] Andrei A. Muchnik, Alexei L. Semenov, and Vladimir Uspensky, *Mathematical metaphysics of randomness*, ***Theoretical Computer Science***, vol. 207 (1998), pp. 263–317.

[14] Andre Nies, ***Computability and randomness***, Oxford University Press, 2009.

[15] C. P. Schnorr, *A unified approach to the definition of random sequences*, ***Mathematical Systems Theory***, vol. 5 (1971), pp. 246–258.

[16] ———, *Process complexity and effective random tests*, ***Journal of Computer and System Sciences***, vol. 5 (1973), no. 3, pp. 378–388.

[17] Claus P. Schnorr, ***Zufälligkeit und Wahrscheinlichkeit: Eine algorithmische Begründung der Wahrscheinlichkeitstheorie***, Lecture Notes in Computer Science #218, Springer Verlag, 1971.

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