IV: Polynomial local search higher in the bounded arithmetic hierarchy

Sam Buss, UCSD
sbuss@math.ucsd.edu

Prague, September 2009

Bounded arithmetic and provably total (multi)functions.

Theory	Graph	(Multi)Function class
S_2^1	Σ_1^b-defined	P functions [B'85]
T_2^1	Σ_1^b-defined	PLS multifunctions. [BK'94]
S_2^k	Σ_k^b-defined	$\mathrm{FP}^{\Sigma_{k-1}^b}$ functions. [B'85]
T_2^k	Σ_k^b-defined	$\mathrm{PLS}^{\Sigma_{k-1}^b}$ multifunctions. [BK'94]
S_2^{k+1}	Σ_k^b-defined	$\mathrm{PLS}^{\Sigma_{k-1}^b}$ multifunctions. [BK'94]
T_2^2	Σ_1^b-defined	Colored PLS. [KST'06]
T_2^k	Σ_1^b-defined	Herbrand analysis [P'03].
"	"	k-turn games, GI_k [ST'ta].
"	"	local improvement. [NST'??].
T_2^k	Σ_i^b-defined	Π_k^b-PLS with Π_{i-1}^b-goal $(1 \leq i \leq k)$ [BB'??,BB'??, this talk]

Π_k^p-PLS — relativizing Polynomial Local Search.

PLS: Recall a *PLS problem* is given by polynomial time cost function c, neighborhood function N, initial function i, and feasible set F.

Gives Σ_1^b-definable functions of T_2^1.

Relativized PLS: $\mathrm{PLS}^{\Pi_k^p} = \mathrm{PLS}^{\Sigma_k^p}$ has F, c, N, i in $P^{\Pi_k^p} = P^{\Sigma_k^p}$.

Gives Σ_{k+1}^b-definable multifunctions of T_2^{k+1}.

New relativization: Π_k^p-PLS has $F \in \Pi_k^p$, but N, c, i are polynomial time.

Also gives Σ_{k+1}^b-definable multifunctions of T_2^{k+1}.

By adding a "goal" property $G(x,s)$ can give the Σ_i^b-definable multifunctions of T_2^{k+1}.

Π_k^p-PLS

Definition

A Π_k^p-PLS problem is given by a predicate $F(x,s) \in \Pi_k^p$, functions $N(x,s)$, $c(x,s)$, $i(x)$ in FP, and a polynomial size bound $d(n)$ that satisfy

(α) $\forall x \forall s (F(x,s) \rightarrow |s| \leq d(|x|))$.

(β) $\forall x (F(x, i(x)))$.

(γ) $\forall x \forall s (F(x,s) \rightarrow F(x, N(x,s)))$.

(δ) $\forall x \forall s (N(x,s) = s \vee c(x, N(x,s)) < c(x,s))$.

and defines a multifunction $f(x) = y$ by:

$$f(x) = y \Leftrightarrow (\exists s \leq 2^{d(|x|)})[F(x,s) \wedge N(x,s) = s \wedge y = (s)_1].$$

PSPACE *algorithm for* Π_k^p-PLS:

Start with $s = i(x)$ and iterate $s \mapsto N(x,s)$

Note: $(s)_1$ is the projection function (Gödel beta function.)

Π^p_k-PLS with Π^p_g-goal G

Definition

A Π^p_k-PLS problem with Π^p_g-goal $G(x,s)$ satisfies the additional property:

(ϵ) $\forall x \forall s(G(x,s) \leftrightarrow [F(x,s) \wedge N(x,s) = s])$.

The graph of the multifunction can now be defined by

$$f(x) = y \;\; \Leftrightarrow \;\; (\exists s \leq 2^{d(|x|)})[G(x,s) \wedge y = (s)_1].$$

Thus, f has a Σ^b_{g+1}-definition.

Definition (Formalized Π^p_k-PLS problems)

For a *formalized* Π^p_k-PLS *problem*, the predicates F and G are given by Π^b_k- and Π^b_g-formulas, N, i, c are polynomial time functions, and the base theory S^1_2 proves conditions (α)-(ϵ).

Note S^1_2, not T^{k+1}_2, proves the conditions.
Formalized Π^p_k-PLS problems are called Π^b_k-PLS problems.

Existence of solutions to Π_k^b-problems

Theorem

Let $\mathcal{P}$ be a Π_k^b-PLS problem. Then T_2^{k+1} proves that, for all x, $\mathcal{P}(x)$ has a solution:

$$T_2^{k+1} \vdash \forall x \exists s (F(x,s) \wedge N(x,s) = s),$$

or, if there is a goal, $T_2^{k+1} \vdash \forall x \exists s (G(x,s))$.

This is a Σ_{k+1}^b- (resp., Σ_{g+1}^b-) definition of a multifunction.

Pf. Similar to before. Σ_{k+1}^b-minimization gives a least c_0 satisfying

$$\exists s \leq 2^{d(|x|)} (c_0 = c(x,s) \wedge F(x,s)).$$

Exact characterization of Σ_i^b-definable functions of T_2^{k+1}.

Theorem

Let $0 \le g \le k$ *and* $A(x,y) \in \Sigma_{g+1}^b$. *Suppose*

$$T_2^{k+1} \vdash (\forall x)(\exists y)A(x,y).$$

Then there is a Π_k^b*-PLS problem* $\mathcal{P}$ *with* Π_g^b*-goal* G *such that* S_2^1 *proves*

$$\forall x \forall s (G(x,s) \to A(x,(s)_1)).$$

Note that the conclusion is provable in S_2^1, but T_2^{k+1} is needed to prove the *existence* of s.

For $k = g = 0$, the Σ_1^b-definable functions of T_2^1 are in PLS.

Why formalization in S^1_2 is important (#1)

Consider a total multifunction defined by $(\forall x)(\exists! y \le t)A(x,y)$, where $A \in \Delta^b_1$. Here is a Π^p_1-PLS search problem for it:

- Initial function: $i(x) = 0$.
- Cost function: $c(x,y) = t - y$.
- Neighborhood function:
$$N(x,y) = \begin{cases} y & \text{if } A(x,y) \text{ or } y \ge t(x) \\ y+1 & \text{otherwise.} \end{cases}$$
- Feasible set: $F(x,y) \;\Leftrightarrow\; (\forall y' < y)(\neg A(x,y')) \wedge y \le t(x)$.
- Goal: $G(x,y) \;\Leftrightarrow\; A(x,y) \wedge y \le t(x)$.

This is a correct Π^p_1-PLS problem independently of provability in T^{k+1}_2. But it is not formalizable in S^1_2, so is not a Π^b_1-PLS problem.

Towards the proof of the "exact characterization" theorem

The proof uses induction on the number of lines a free-cut free T_2^{k+1}-proof to establish a witnessing lemma. W.l.o.g. all formulas are $s\Sigma_{k+1}^b$.

Def'n: Let $A(\vec{c}) \in s\Sigma_{k+1}^b$. Recall that $Wit_A(\vec{c}, u)$ is a Π_k^b-formula that states u is value for the outermost existential quantifier of $A(\vec{c})$ making $A(\vec{c})$ true.

For a sequent $\Gamma \longrightarrow \Delta$, where Γ is $A_1, \ldots, A_k$ and Δ is $B_1, \ldots, B_\ell$,

$$Wit_\Gamma(\vec{c}, u) \text{ is } \bigwedge_{i=1}^{k} Wit_{A_i}(\vec{c}, (u)_i).$$

and

$$Wit_\Delta(\vec{c}, u) \text{ is } \bigvee_{j=1}^{\ell} \left((u)_1 = j \wedge Wit_{B_j}(\vec{c}, (u)_2)\right).$$

Theorem (Witnessing Lemma.)

If T_2^{k+1} proves a sequent $\Gamma \longrightarrow \Delta$ of $s\Sigma_{k+1}^b$-formulas with free variables $\vec{c}$, then there is a multifunction f defined by a Π_k^b-PLS problem such that

$$S_2^1 \ \vdash \ Wit_{\bigwedge \Gamma}(\vec{c}, u) \wedge y = f(\langle \vec{c}, u \rangle) \rightarrow Wit_{\bigvee \Delta}(\vec{c}, y).$$

The proof is by induction on length of a free-cut free proof. All lines in the proof are sequents of $s\Sigma_{k+1}^b$-formulas.

The arguments split into cases based on the final inference of the T_2^{k+1}-proof P. Since PLS functions are easily seen to be closed under polynomial time operations, many of the arguments are similar to earlier witnessing arguments. (But not all all!)

Suppose the last inference in P is $\forall \leq$:right.

$$\frac{b \leq t, \Gamma \longrightarrow \Delta, A(\vec{c}, b)}{\Gamma \longrightarrow \Delta, (\forall x \leq t) A(\vec{c}, x)}$$

Let $f(\vec{c}, u)$ be given by the induction hypothesis for witnessing the upper sequent.

As before, the idea is to set $\mu_{\neg A}(\vec{x})$ equal to the least $b \leq t$ s.t. $\neg A(\vec{c}, b)$, or equal to $t+1$ if no such b exists.

In the former case, $g(\vec{c}, u) = f(\vec{c}, b, \langle 0 \rangle * u)$, where f is given by the induction hypothesis. In the latter case, $g(\vec{c}, u) = \langle \ell, 0 \rangle$.

However, f and g do not have access to any oracle for Π^b_k, and can use only polynomial time operations. For this reason, we must define a special Π^b_k-PLS algorithm, called $\mathcal{P}_A$ that (in essence) computes $\mu_A(\vec{c})$ for A a Σ^b_k-formula.

Lemma

Let $A(x) = (\exists y{\leq}t)B(y,x) \in s\Sigma^b_k$. *There is a* Π^b_k*-PLS problem* $\mathcal{P}_A$ *that determines the truth of* $A(x)$ *by computing*

$$\mathcal{P}_A(x) = \begin{cases} \langle 0, t+1\rangle & \textit{if } \neg A(x) \\ \langle 1, i\rangle & \textit{if } i{\leq}t \textit{ is the least value s.t. } B(i,x). \end{cases}$$

Pf. Define initial function $i(x) := \langle 0,0\rangle$. Define

$$N(x,\langle 0,i\rangle) = \begin{cases} \langle 0, i+1\rangle & \text{if } \neg B(i,x),\ i \leq t. \\ \langle 1, i\rangle & \text{if } B(i,x),\ i \leq t \end{cases}$$

$$N(x,s) = s \text{ for all other } s.$$

For $k > 1$, determining $\neg B(i,x)$ involves calling $\mathcal{P}_{\neg B}$, a Π^b_{k-1}-PLS problem. For $k = 0$, it is polynomial time to decide $B(i,x)$.

Feasible set is $F(x,\langle 0,i\rangle) \Leftrightarrow i{\leq}t+1 \wedge (\forall j{<}i)(\neg B(j,x))$ and
$F(x,\langle 1,i\rangle) \Leftrightarrow i{\leq}t \wedge B(i,x) \wedge (\forall j{<}i)(\neg B(j,x))$.

Cost function $c(x,\langle j,i\rangle) = t+1-i-j$. $\square$

Skolemization: A stronger version of Π_k^b-PLS witnessing

Skolemization: For a Boolean combination of formulas, create equivalent prenex form by the following procedure. Find all outermost blocks of quantifiers not yet processed. Bring out all universal ones first, then all existential ones. Repeat until in prenex form. Then Skolemize with terms.

Example: Recall (γ) is: $\forall x, s(F(x,s) \to F(x, N(x,s)))$

Suppose F is $\forall y \exists z F_0(y,z)$. Then (γ) is Skolemized as follows:

Prenex form: $\forall x, s, y_2 \exists y_1 \forall z_1 \exists z_2 (F_0(x,s,y_1,z_1) \to F_0(x,s,y_2,z_2))$.

Skolem form:
$\forall x, s, y_2, z_1 (F_0(x,s,r(x,s,y_2),z_1) \to F_0(x,s,y_2,t(x,s,y_2,z_1)))$.

where r and t are terms (over the language $0, S, +, \cdot, \dot{-}, MSP$ that allows simple fixed-length sequence coding.) r and t are polynomial time.

Definition

A Π_k^b-PLS problem with Π_g^b goal is *formalized in Skolem form* provided the functions N, c, and i are defined by *terms*, and the formulas F and G are strict formulas in $s\Pi_k^b$ and $s\Pi_g^b$, and provided S_2^1 proves all the conditions (α)-(δ) plus

(ϵ') $\forall x \forall s (G(x,s) \rightarrow [F(x,s) \wedge N(x,s) = s])$

(ϵ'') $\forall x \forall s ([F(x,s) \wedge N(x,s) = s] \rightarrow G(x,s))$

in Skolem form using terms as Skolem functions.

Trivially:

Theorem

If $\mathcal{P}$ is formalized in Skolem form, it is also formalized in the usual form.

Exact characterization revisited, Skolemized form

Theorem

Let $0 \leq g \leq k$ and $A(x,y) \in \Sigma^b_{g+1}$. Suppose

$$T_2^{k+1} \vdash (\forall x)(\exists y)A(x,y).$$

Then there is a Π^b_k-PLS problem $\mathcal{P}$ with Π^b_g-goal G which is formalized in Skolem form such that S^1_2 proves a Skolemization of:

$$\forall x \forall s(G(x,s) \to A(x,(s)_1)).$$

The proof of the theorem is similar to before, but much more delicate.

One potential problem: For $A \in \Pi_k^b$, the formula

$$A \rightarrow A \wedge A$$

may not be provable in Skolem form by S_2^1.

We need this, however, in many cases, especially ones with (explicit or implicit) contraction in the antecedent.

Solution: Use $\mathcal{P}_A$, the Π_k^b-PLS problem that determines the truth of A. The formula

$$A(x) \wedge y = \mathcal{P}_A(x) \rightarrow A(x) \wedge A(x)$$

is provable in Skolem form by S_2^1.

A separation conjecture

We can set up a "generic" Skolemized Π_k^b-PLS problem with Π_0^b-goal as follows:

Adjoin a new predicate symbol for G and a new predicate symbol F_0 for the sharply bounded subformula of F.
Also adjoin new function symbols which are used as Skolem functions for the Π_k^b-PLS problem's defining conditions.

Then, the Skolemized definition of the Π_k^b-PLS problem can be expressed as a single $\forall\Delta_0^b$-formula.

Encoding the new functions and predicates by a single new predicate α, we can encode this $\forall\Delta_0^b$-formula as a single $\forall\Delta_0^b$-formula $\forall x \Psi(x, \alpha)$.

Consider the formula

$$\forall x \Psi(x) \rightarrow \forall x \exists y \leq x (y = N(x, y) \wedge G(x, y)). \qquad (1)$$

By the relativized version of the first theorem, it is provable in $T_2^{k+1}(\alpha)$.

On the other hand, by the conjectured properness of the bounded arithmetic and polynomial time hierarchies, we expect this is not provable in $T_2^k(\alpha)$.

This gives a single $\forall\Sigma_1^b(\alpha)$-formula that is known to be provable in $T_2^{k+1}(\alpha)$ but conjectured to not be provable by $T_2^k(\alpha)$.

Why formalization in S_2^1 is important (#2). Since the Skolem functions are polynomial time, they can be conservatively added to S_2^1, T_2^{k+1}, etc., and can be used freely in induction axioms. Thus, it is reasonable to allow $T_2^{k+1}(\alpha)$ use α freely in induction axioms.

Conjectured separation for constant depth Frege proofs

By Paris-Wilkie translation, we get a conjectured separation for bounded depth propositional proof systems as follows:

A *depth k Tait system* has sequents of formulas of depth k. Depth is measured by alternations of $\wedge$'s and $\vee$'s with small fanin at the bottom level counting as a $\frac{1}{2}$ depth.

The $T_2^{k+1}(\alpha)$-proof (1) translates to a depth $k-1$ proof by the Paris-Wilkie translation (after several careful transformations, as discussed in the last talk, and note the extra $\frac{1}{2}$ reduction in depth for this special case).

This gives a family Ξ_k of sets of sequents of literals such that members of Ξ_k have depth $k-1$ Tait-style refutations, but are conjectured to not have depth $k-1\frac{1}{2}$ depth refutations (by the conjectured non-provability of (1) in T_2^k).

Some open problems

1. Is there a non-uniform version of the witnessing theorems for T_2^k that will apply to depth $k - 1\frac{1}{2}$ propositional proofs?
2. Are there good analogues of Thms 2 or 3 for fragments of Peano arithmetic?
3. The weak pigeonhole principle (WPHP) for $\alpha : [2n] \to [n]$ is provable in $T_2^2(\alpha)$, but not $T_2^1(\alpha)$. (Paris-Wilkie-Woods, 1988; Maciel-Pitassi-Woods, 2000.) Can this be reversed?
4. Ramsey's Theorem for pairs is provable in $I\Delta_0 + \Omega_1$ (perhaps S_2^3?). (Pudlak, 1991.) Can this be reversed?

Some selected references

- S. Buss, *Bounded Arithmetic*, Ph.D. thesis, 1985. Bibliopolis, 1986. Also available online.
- S. Buss, J. Krajíček, An application of Boolean complexity to separation problems in bounded arithmetic. Proc. LMS 69 (1994) 1-21.
- P. Pudlák, Consistency and games — in search of combinatorial principles, Logic Colloquium, 2003.
- J. Krajíček, A. Skelley, N. Thapen, "NP search problems in low fragments of bounded arithmetic", *JSL* 72 (2007)649-672.
- A. Skelley, N. Thapen, The provably total search problems of bounded arithmetic, ta.
- P. Nguyen, A. Skelley, N. Thapen, The provably total NP search problems of weak second order bounded arithmetic, submitted, 2009.
- A. Beckmann, S. Buss, Characterizing Definable Search Problems in Bounded Arithmetic via Proof Notations, submitted, 2009.
- A. Beckmann, S. Buss, Polynomial Local Search in the Polynomial Hierarchy and Witnessing in Fragments of Bounded Arithmetic, submitted, 2009.