

Bounded Arithmetic and a Consistency Result for NEXP vs P/poly

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$$L \subseteq NL = \text{coNL} \subseteq P \subseteq NP \subseteq (N)PSPACE \subseteq EXP \subseteq NEXP$$

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“N” = “nondeterministic”

“P” = “polynomial (time)”

“EXP” = “exponential time”

“PH” = “polynomial time hierarchy”

“P/poly” = “p-time + polynomial advice; i.e., polynomial size circuits”

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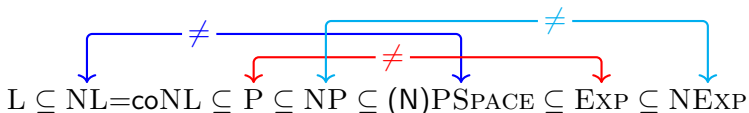
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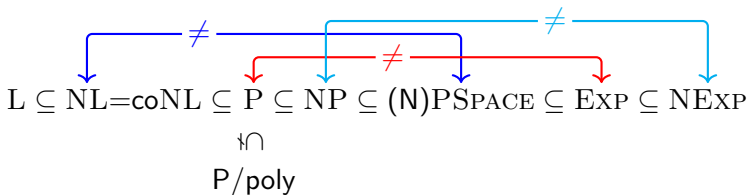
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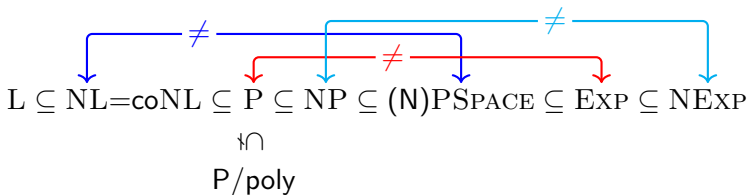
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Thm: $NP \subset P/\text{poly} \Rightarrow \text{PH} \downarrow = \Sigma_2^P$. [Karp-Lipton '82]

Thm: $\text{EXP} \subset P/\text{poly} \Rightarrow \text{EXP} = \text{PH} = \Sigma_2^P = \text{MA}$. [Meyer; BFL'91]

Thm: $\text{NEXP} \subset P/\text{poly} \Leftrightarrow \text{NEXP} = \text{PH} = \Sigma_2^P = \text{MA}$. via Easy Witness Thm [IKW '02]

Thm: $\text{NEXP} \not\subset \text{ACC}^0$. [Williams '14]

This talk:

“NExp $\not\subset$ P/poly” is consistent with the bounded arithmetic theory V_2^0 .

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• ORACLE SEPARATIONS •

First: an oracle separation:

Theorem: There is also an oracle Ω such that $P^\Omega \neq NP^\Omega$.
[Baker-Gill-Solovay'75]

Can be recast as:

Theorem: There is an oracle Ω so that $NP^\Omega \not\subseteq P^\Omega/\text{poly}$.

Further: there is an Ω so that $NEXP^{\Omega[\text{poly}]} \not\subseteq P^\Omega/\text{poly}$.

There is an oracle such that $NEXP^{\Omega[\text{poly}]} = P^\Omega$.

Moral: Separation proofs have to use non-relativizing techniques.

Disadvantage: Relativization.

• NATURAL PROOFS •

[Razborov-Rudich'97]

A proof of $\mathcal{C} \notin P/\text{poly}$ is “**natural**” if it is

- Useful (Effective)
- Constructive
- Large (applies to many Boolean functions)

Theorem: There are no natural proofs that $NP \not\subseteq P/\text{poly}$ if a (generally believed) strong pseudorandom number generator (SPRNG) conjecture holds. [RR'07]

Natural proofs operate on truth tables to identify Boolean functions that require large circuits.

Disadvantage: The result is conditional on SPRNG.

• ALGEBRIZATION •

[Fortnow'94; Aaronson-Wigderson'08; Impagliazzo-Kabanets-Kolokolova'09]
Work with “algebrizing oracles — Boolean oracles Ω and their extensions $\tilde{\Omega}$ to low-degree polynomials.

Theorem: [AW'08]

- $IP = PSPACE$ (e.g.) has an algebrizing proof.
- $NP \subset P/poly$ and $NEXP \subset P/poly$ cannot be proved with algebrizing techniques.
E.g. for some Ω , $NEXP^{\tilde{\Omega}[\text{poly size}]} \not\subset P^{\Omega}/poly$.

Moral: Separation proofs have to use non-algebrizing techniques.

Disadvantage: Relativization.

Part II: Quick review of witness circuits

Witnessing for NP

Let $Q(x) \Leftrightarrow (\exists y \leq t(x))P(x, y)$ be an NP predicate.

Here, $P(\cdot, \cdot)$ is p-time and $t(x)$ is poly-growth rate.

A **witness circuit** for $Q(x)$ is a multi-output Boolean circuit $D(x)$ such that $\forall x$,

$$Q(x) \Leftrightarrow P(x, D(x)).$$

I.e. $(\forall x \leq b)(\forall y \leq t(b))[P(x, y) \rightarrow P(x, D(x))]$.

Theorem

If NP has polynomial-size circuits ($\text{NP} \subset \text{P/poly}$), then NP has polynomial-size witness circuits.

Proof idea: $D(x)$ uses poly-size subcircuits to query the bits of a minimal y one at a time.

— — —

The property of being a witness circuit is Π_1^b . With $Q := \text{SAT}$, this can be exploited to prove the Karp-Lipton theorem.

Witnessing for NEXP

Let $Q(x) \Leftrightarrow (\exists^2 X \leq 2^{p(|x|)})P(x, X)$ be an NEXP predicate.

Here,

$X \in \{0, 1\}^{2^{p(|x|)}}$ - an exponentially long bit string (or, oracle)

and

$P(x, X) \in \text{EXP} := \text{TIME}(2^{q(|x|)})$

p, q are polynomials.

Easy Witness Theorem: [Impagliazzo-Kabanets-Wigderson'02]

Suppose $\text{NEXP} \subset \text{P/poly}$. Then there are polynomial size circuits $D(\cdot)$ so that, for all x ,

$$(\exists^2 X \leq t(x))P(x, X) \Leftrightarrow P(x, D(x)).$$

That is, $D(x) := D(x, i)$ outputs the value of $X(i)$.

III. Theories of Arithmetic

Results reported in this talk:

- Describe second-order fragments of bounded arithmetic, including V_2^i , $i \geq 0$.
- Formulate “ $\text{NEXP} \not\subseteq \text{P/poly}$ ” as second-order formula. Two forms are formulated.
- Prove that $\text{NEXP} \subset \text{P/poly}$ is not provable in V_2^0 .
Equivalently: $\text{NEXP} \not\subseteq \text{P/poly}$ is consistent with V_2^0 .
Equivalently: $\text{NEXP} \not\subseteq \text{P/poly}$ is true in some model of V_2^0 .
- Sketch of the proof.
and
- A “hardness magnification” lifting hardness for $S_2^1(\alpha)$ to hardness for $V_2^1(\alpha)$

Part III. Theories of Bounded Arithmetic (subtheories of PRA)

PV		Equational
\cap		
$S_2^1 \subseteq T_2^1 \subseteq S_2^2 \subseteq T_2^2 \subseteq \dots \subseteq T_2$	$:= \bigcup_i T_2^i$	First-order
	$\nabla \cap$	
	$V_2^0 \subseteq V_2^1 \subseteq V_2^2 \subseteq \dots$	Second-order

All theories include second-order objects X (essentially oracles).

PV & S_2^1 - Theories for polynomial time. [Cook'75; B'86]

T_2^i - Theories for the levels of the polynomial time hierarchy (PH). [B'86]

V_2^1 - Theory for exponential time. [B'86]

Language for bounded arithmetic:

Basic functions: $0, S, +, \cdot, \#, \lfloor \frac{1}{2}x \rfloor, <$.

Polynomial time functions. Every p -time function (and relation).

First-order variables and quantifiers. $\forall x, \exists x$ - range over integers.

Second-order variables and quantifiers. $\exists^2 X, \forall^2 X$ - range over (finite) sets of integers, i.e., over “oracles” or exponentially long binary strings.

Axioms for bounded arithmetic:

Defining axioms for basic functions and p -time symbols.

Boundedness and Extensionality for second-order objects.

Induction/Minimization for second-order objects.

Length-induction (PIND/LIND) or usual induction (IND).

Comprehension for some class Φ of formulas.

V_2^0 has $\Sigma_0^{1,B}$ -comprehension. Essentially PH-comprehension.

The theory T_2 can formulate many complexity results:

- Cook-Levin Theorem. [Cook'75; B'86]
- Karp-Lipton Theorem. [B'86]
- Hastad Switching Lemma. [Razborov'95]
- $\text{PARITY} \notin \text{AC}^0$. [Krajíček'95]
- Rabin test for primality. [Jeřábek'04]
- $\text{BPP} \in \text{P/poly}$ [Jeřábek'04]
- $\text{BPP} \in \Sigma_2^P \cap \Pi_2^P$ [Jeřábek'07]
- $\text{MA} = \text{MAM}$ (Merlin-Arthur). [Jeřábek'07]
- PCP Theorem [Pich'15]
- and more ...

Prior Consistency Results (selected)

Razborov'95: If the SPRNG conjecture holds, S_2^2 cannot prove (slightly) superpolynomial lower bounds on circuit size.

Theorem: [Cook-Krajíček'07]

- If $PH \not\subseteq P^{NP[\log]}$, then $NP \not\subseteq P/\text{poly}$ is consistent with S_2^1 .
- If $PH \not\subseteq P^{NP}$, then $NP \not\subseteq P/\text{poly}$ is consistent with S_2^2 .

Theorem: [Krajíček-Oliviera'17],[Carmosino-Kabanets-Kolkolova-Olviera'21]

For fixed c ,

- $NP \not\subseteq \text{SIZE}(n^c)$ is consistent with S_2^1 .
- $P^{NP} \not\subseteq \text{SIZE}(n^c)$ is consistent with S_2^2 .
- $ZPP^{NP} \not\subseteq \text{SIZE}(n^c)$ is consistent with APC_2 .

[Bydövký-Müller'20], [Bydövký-Krajíček-Müller'20], [Pich'15],

[Pich-Santhanan'21], [Li-Oliviera'23] have other unconditional independence results.

For example,

Theorem: [Pich-Santhanan'21] For $\delta < 1$

- It is consistent with PV and $T_{APC_1}^0$ that NP-predicates cannot be approximated by co-nondeterministic circuits of size $2^{\delta n}$.

These proofs nearly all use the KPT version of the Herbrand witnessing theorem. Some of them use the randomization technique of the Nisan-Wigderson theorem [Nisan-Wigderson'94], extending [Krajíček'12].

Part IV: Formalizations of $\text{NEXP} \not\subseteq \text{P/poly}$

Let $M(x)$ be a canonical NEXP -complete predicate.

Formalization #1: For each $c \in \mathbb{N}$, let α^c be the formula

$\forall 2^n \exists \text{circuit } C < 2^{n^c} \forall x < 2^n [$

$C(x) = 1 \rightarrow \exists^2 Y (Y \text{ codes an accepting computation of } M(x)) \wedge$

$C(x) = 0 \rightarrow \neg \exists^2 Y (Y \text{ codes an accepting computation of } M(x))]$

- n is a size parameter.
- Inputs x are strings of length n .
- C ranges over Boolean circuits of size $\approx n^c$.
- $C(x) = 1 \Leftrightarrow M$ accepts x .

Part IV: Formalizations of $\text{NEXP} \not\subseteq \text{P/poly}$

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Formalization #2: For each $c \in \mathbb{N}$, let β^c be the formula

$$\forall 2^n \exists \text{circuits } C, D < 2^{n^c} \forall x < 2^n [\\ C(x) = 1 \rightarrow (D(x, \cdot) \text{ codes an accepting computation of } M(x)) \wedge \\ C(x) = 0 \rightarrow \neg \exists^2 Y (Y \text{ codes an accepting computation of } M(x))]$$

$\bigvee_c \alpha^c$: Exactly states “ $\text{NEXP} \subset \text{P/poly}$ ”.

$\bigvee_c \beta^c$: Equivalent to “ $\text{NEXP} \subset \text{P/poly}$ ” by Easy Witness Lemma.

$\{\neg\alpha^c\}_{c \in \mathbb{N}}$: Exactly states “NEXP $\not\subseteq$ P/poly”.

$\{\neg\beta^c\}_{c \in \mathbb{N}}$: Equivalent to “NEXP $\not\subseteq$ P/poly”
via Easy Witness Lemma.

The implications $\beta_c \rightarrow \alpha_c$ are trivial
(via comprehension on $\{y : D(x, y)\}$).

Theorem (Atserias-B.-Müller'23)

- $V_2^0 + \{\neg\alpha^c\}_{c \in \mathbb{N}}$ is consistent.
- $V_2^0 + \{\neg\beta_c\}_{c \in \mathbb{N}}$ is consistent.

I.e., $V_2^0 +$ “NEXP $\not\subseteq$ P/poly” is consistent.

Proof sketch

Proof is by contradiction.

- Suppose $V_2^0 \models \alpha^c$ for some $c \in \mathbb{N}$.
(For sake of a contradiction.)
- We'll show that V_2^0 proves PHP_n^{n+1} in this case.

$\text{PHP}_x^{x+1} :=$ Pigeonhole principle on x many pigeons.

- But this is impossible, because the Paris-Wilkie translation would then imply that there are quasipolynomial size, constant-depth Frege proofs of PHP_n^{n+1} . These are known not to exist, [Beame-Impagliazzo-Krajíček-Pitassi-Pudlák-Woods'92]
- In second-order arithmetic, the statement $\neg \text{PHP}_x^{x+1}$ can be expressed as

$$\exists^2 Z [\quad \forall u \leq x (Z(u) < x) \wedge \\ (\forall u < v \leq x)(Z(u) \neq Z(v))]$$

- Note that $\neg\text{PHP}_x^{x+1}$ is a NEXP-predicate.
- Since we suppose $V_2^0 \models \alpha^c$, there is a family of polynomial size Boolean circuits $C_n(x)$ such that $C_{|x|}(i)$ outputs *True* iff there is a Z violating the pigeonhole principle PHP_i^{i+1} (for $i \leq x$).
- Then, similar to the Cook-Rechhow [’79] proof of PHP, this allows V_2^0 to prove the pigeon hole principle holds for all x . Namely, from a Z violating PHP_i^{i+1} , it is easy to construct (in V_2^0) a Z' violating PHP_{i-1}^i .
- From this, induction — on the values of $C_{|x|}(i)$ — allows V_2^0 to prove $\forall x \neg\text{PHP}_x^{x+1}$
- This gives the desired contradiction.

A similar proof gives a stronger result:

Theorem (Atserias-B.-Müller’23)

$V_2^0 + \text{“NEXP} \not\subseteq \text{PH/poly”}$ is consistent.

Theorem (Atserias-B.-Müller'23)

For the $\{\neg\beta^c\}$ formalization:

- If $S_2^1 \not\vdash \text{NEXP} \not\subseteq \text{P/poly}$, then $V_2^1 \not\vdash \text{NEXP} \not\subseteq \text{P/poly}$.
- If $V_2^1 \vdash \text{NEXP} \not\subseteq \text{P/poly}$, then $S_2^1 \vdash \text{NEXP} \not\subseteq \text{P/poly}$.

This is an intriguing result since the theory V_2^1 is so strong.

Indeed, Razborov['95] identifies V_2^1 as a strong theory for which independence results will be highly indicative.

Proof sketch:

A model \mathcal{M} of $S_2^1 + \beta^c$ can be enlarged to be a model \mathcal{N} of $S_2^1 + \beta^c$ plus $\exists^2\Pi_1^b$ -comprehension for formulas without free second-order parameters. Namely, by taking the second-order objects of \mathcal{N} to be those definable by 2^{n^c} -size circuits in \mathcal{M} . This is also a model of $V_2^1 + \beta^c$.

Open Questions

- Is $V_2^0 + \neg\alpha^{\log \log x}$ consistent? (Or, slower growing value for c ?)
- Is $V_2^0 + \text{“EXP} \not\subset \text{P/poly”}$ consistent?
Is $V_2^0 + \text{“PSPACE} \not\subset \text{P/poly”}$ consistent?
- Is $V_2^0 + \text{“NP} \subset \text{P/poly”}$ consistent?
- Do V_2^0 or V_2^1 prove the Easy Witness Lemma?
- Independence results for V_2^1 ?

Thank you!