Math 160A - Winter 2002
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Theorem Sequence[*]

Definition. Σ_1 and Σ_2 are *tautologically equivalent* if for all ϕ, $\Sigma_1 \vDash \phi$ iff $\Sigma_2 \vDash \phi$. $\Sigma_1 \vDash \Sigma_2$ if, for $\phi \in \Sigma_2$, $\Sigma_1 \vDash \Sigma_2$.

Theorem. The following are equivalent.

(1) Σ_1 and Σ_2 are tautologically equivalent.

(2) $\Sigma_1 \vDash \Sigma_2$ and $\Sigma_2 \vDash \Sigma_1$.

(3) For all truth assignments v, v satisfies Σ_1 iff v satisfies Σ_2.

Theorem D. If Σ is a finite set of wffs, then Σ is tautologically equivalent to a singleton set.

Definition. Let Σ and Γ be sets of wffs. Then, Σ and Γ are *complementary* if for all truth assignments v, v satisfies Σ iff v does not satisfy Γ.

Theorem E. Suppose Σ and Γ are complementary set of wffs and that Σ is a finite set. Then Γ is tautologically equivalent to a singleton set.

Theorem F. Suppose Σ and Γ are complementary set of wffs. Then Σ and Γ are each tautologically equivalent to a singleton set.

Lemma G. Suppose Σ and Γ are complementary set of wffs. Then $\Sigma \cup \Gamma$ is unsatisfiable.

Theorem H. There are wffs ϕ and ψ such that $\exists x\phi \wedge \exists x\psi \nvDash \exists x(\phi \wedge \psi)$.

Theorem I. $\vDash \exists x(\exists y P(y) \rightarrow P(x))$.

Theorem J. $\vDash \forall x(P(x) \rightarrow \exists y P(y))$.

[*]This is not a traditional Moore-method theorem sequence, in that it does not make by itself a coherent course. Rather, it is the actual sequence of theorems assigned to students in conjunction with a course a based on Enderton's textbook, *A Mathematical Introduction to Logic*.

Theorem K. $\not\models \forall x(P(x) \leftrightarrow \exists y P(y))$.

Theorem L. $\forall x(P(x) \vee Q(x)) \not\models \forall x P(x) \vee \forall x Q(x)$.

Theorem M. Prove the following. It is enough to prove the existence of the deductions, you do not need to explicitly write out the complete deduction.

a. $\vdash \forall x(\alpha \to \beta) \to (\alpha \to \forall x \beta)$, where x is not free in α.

b. $\vdash x = y \to \forall z(Pxz \leftrightarrow Pyz)$.

c. $\vdash \forall x(\alpha \to \beta) \leftrightarrow (\exists x \alpha \to \beta)$, where x is not free in β.

Theorem N. Give examples of formulas α and β such that

$$\not\models \forall x(\alpha \to \beta) \leftrightarrow (\alpha \to \forall x \beta).$$

Lemma O. Prove that if Γ is consistent, then at one of $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$ is consistent.

Theorem P. Prove the consistency of the set Δ constructed in class for the third step in the proof of the completeness theorem.