# UNPREDICTABILITY

whether or not a sequence is generated randomly.

* References:

- Paper of Merkle, et.al.
- Book - Downey, Hirschfeld

**Def.** A (monotonic) stake function:

$$q : \{0,1\}^* \to [0,2]$$

interpreted as: if $q(\sigma) < 1$, bet that the next bit is 1

$q(\sigma) > 1$, bet 0

$q(\sigma) = 0$, agnostic.

**Def.** A payoff function, $C_q : \{0,1\}^* \to [0,2]$, defined by

$$C_q(\sigma 0) = q(\sigma)$$

$$C_q(\sigma 1) = 2 - q(\sigma)$$

**Def.** The capital function, $d_q : \{0,1\}^* \to \mathbb{R}^{\geq 0}$, defined by

$$d_q(\sigma_0, \sigma_1, \dots \sigma_n) = C_q(\sigma_0) C_q(\sigma_0 \sigma_1) \cdots C_q(\sigma_0 \cdots \sigma_n)$$

☆ Convention: initial capital, $d(\varepsilon) = 1$

**Def.** Martingales, $d : \{0,1\}^* \to \mathbb{R}^{\geq 0}$ s.t. $\forall \sigma \in \{0,1\}^*$,

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2} \qquad \text{where } d(\sigma) \text{ is capital}$$

These represent a fair game.

**Comment.** Martingales $=^*$ capital fcn of monotonic stake function, i.e., (i) given a Martingale $d$, $\exists q$ s.t. $d_q = d$, (ii) and given $q$, $d_q$ is a Martingale.

Proof of (ii): ( Given $q$, check $d_q(\sigma) = \frac{d_q(\sigma 0) + d_q(\sigma 1)}{2}$ )

$$d_q(\sigma 0) = \underbrace{C_q(\sigma_0) C_q(\sigma_0 \sigma_1) \cdots C_q(\sigma)}\, C_q(\sigma 0)$$

$$= \quad d_q\, C_q(\sigma 0) = d_q(\sigma)\, q(\sigma)$$

(similarly)

$$d_q(\sigma 1) = \quad d_q\, C_q(\sigma 1) = d_q(\sigma)(2 - q(\sigma))$$

so, $\frac{d_q(\sigma 0) + d_q(\sigma 1)}{2} = \frac{d_q(\sigma) q(\sigma) + d_q(\sigma)(2 - q(\sigma))}{2} = d_q(\sigma)$ ✓

(i): Define $q(\sigma) = \frac{d(\sigma 0)}{d(\sigma)}$

Can check $q(\sigma) = 2 - \frac{d(\sigma 1)}{d(\sigma)}$ b/c Martingale

Def. Let $X \in \{0,1\}^\omega$, $q : \{0,1\}^* \to [0,2]$, then the payoff while playing on $X$ is defined to be $C_q^X : \mathbb{Z}^+ \to [0,2]$,

$$C_q^X(n+1) = \begin{cases} q(X \upharpoonright n+1) & \text{if } X(n+1) = 0 \\ 2 - q(X \upharpoonright n+1) & \text{if } X(n+1) = 1 \end{cases}$$

where $X \upharpoonright n+1 = X(0) X(1) \cdots X(n) \sqcup$

And the capital function for this game becomes $d_q^X : \mathbb{Z}^+ \to \mathbb{R}^{\geq 0}$,

$$d_q^X(n) = \prod_{i=1}^{n} C_q^X(i)$$

Def. The Capital fcn / Martingale succeeds on $X$ if

$$\limsup_{n \to \infty} d_q^X(n) = \infty$$

Sanity Check

$\limsup_{n \to \infty} d_q^X(n) = \infty \Rightarrow \exists^\infty n,\ C_q^X(n) > 1$

i.e., $\exists^\infty n \big( q(X \upharpoonright n) > 1$ and $X(n+1) = 0$

or $q(X \upharpoonright n) < 1$ and $X(n+1) = 1 \big)$

Fact. $\forall X \in 2^\omega$, $\exists$ a Martingale that succeeds on it.

$\boxed{\text{The gist: If there is a Martingale that succeeds on } X \text{, and is effective then } X \text{ is not "random."}}$