Math267 - Introduction to Set Theory

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Part I

Homework Problems

1. Assume y is a non-empty set. Prove that $\bigcap y$ exists.

2. The crucial property of ordered pairs is that they have unique first and second elements and are determined by these elements. Prove that $(x, y) = (u, v)$ iff $x = u$ and $y = v$.

3. Suppose someone has proposed

$$[x, y] \ := \ \{x, \{x, y\}\}$$

as an alternative definition of ordered pair. Show that the property problem #2 hold for this definition too. (You may use the Axiom of Foundation for this problem.)

4. You need to use the Axiom of Foundation for this problem. Prove that $x \notin x$.

5. Let w be a set.

a. Let f be a set of order pairs which is a function. Write out carefully the definition of $z = f[w]$.

b. Let $\boldsymbol{F}$ be a function defined by a formula $\varphi(x, y)$. Write out carefully the definition of $\boldsymbol{F}[w]$.

6. Prove that AC is equivalent to AC'.

7. Prove that the well-ordering principle WO implies the Axiom of Choice AC.

8. (Harder) Prove that AC implies WO.

9. Suppose x and y are transitive. For each of the following assertions, either provide a proof or provide a counterexample.

a. $x \subset \wp(x)$.

b. $\wp(x)$ is transitive.

c. $x \cup y$, $x \cap y$, and $x \setminus y$ are transitive.

d. If $z \in x$, then z is transitive.

10. Prove the following, for all $x \in \omega$:

a. If $0 \neq x$, then $0 \in x$.

b. If $y \in x$, then $S(y) \in x$ or $S(y) = x$.

c. $x \subset \omega$.

11. Prove: For all x and y in ω, either $x \in y$, $x = y$ or $y \in x$.

12. Prove that there is no infinite sequence $(x_0, x_1, x_2, \ldots)$ such that $x_{i+1} \in x_i$ for all i. (Use the Axiom of Foundation.)

13. Write formulas with define the functions $x \cdot y$ and x^y over the natural numbers. Convince yourself of the existence and uniqueness conditions for your definitions; and that symbols for multiplication and exponentiation on the natural numbers can be used without exceeding the expressive power of the first-order language of set theory.

14. Do the same for the Fibanacci numbers defined by $F(0) = F(1) = 1$ and $F(n+2) = F(n) + F(n+1)$.

15. Prove that (the axioms of set theory imply that) integer addition is commutative and associative. (Hints are available.)

16. Prove that ordinal addition is associative: i.e., $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

17. Prove that the Hausdorff Maximal Principle is equivalent to the Axiom of Choice.

18. Prove that $\alpha \cdot \beta$ is the otype of the cross-product $\alpha \otimes \beta$.

19. Prove:

a. Each of $\alpha+\beta$, $\alpha\cdot\beta$ and α^β are strictly monotonically increasing as functions of β. (For multiplication, you must assume $\alpha>0$, and for exponentiation, you must assume $\alpha>1$.)

b. Each of $\alpha+\beta$, $\alpha\cdot\beta$ and α^β are non-decreasing as functions of α. For exponentiation, you must assume $\alpha>0$.

20. Give a natural characterization (or "re-definition") of the order type α^β, similar in spirit to our characterizations of addition and multiplication in terms of concatenation and crossproduct.

21. Prove that if α and β are countable, then so are $\alpha+\beta$, $\alpha\cdot\beta$ and α^β. (There is no need to re-prove facts like the union of countably many sets is countable.)

22. Prove that ordinal arithmetic is left distributive. Give an example of how right distributivity can fail.

23. Prove the following about ordinal exponentiation.

a. $\alpha^{\beta+\gamma}=\alpha^\beta\cdot\alpha^\gamma$.

b. $(\alpha^\beta)^\gamma=\alpha^{\beta\cdot\gamma}$.

24. Prove the following:

a. For all α, $\omega^{\epsilon_\alpha}=\epsilon_\alpha$.

b. Prove that if $\omega^\beta=\beta$, then $\beta=\epsilon_\alpha$ for some α.

c. Prove that ϵ_α is the α-th fixed point of the map $\beta\mapsto\omega^\beta$. (Or at least make sense of this statement and convince yourself it is true.)

d. Prove that there is an ϵ-number α so that $\epsilon_\alpha=\alpha$.

25. (AC) Prove that a set is Dedekind finite iff it is finite.

26. Without using the Axiom of Choice, prove that for all cardinals κ, there is a cardinal $\lambda>\kappa$. [Hint: consider the supremum of the otypes of the well-orderings of κ.]

27. Suppose that $f: ON\to ON$ is a strictly increasing and continous function (i.e., $f(\sup A)=\sup f[A]$ for all sets of ordinals A.) Prove that there exist arbitrarily large fixed points of f. Prove that for all κ, there is an $\alpha>\kappa$ such that $\aleph_\alpha=\alpha$.

28. Give an example of cardinals α and β such that there is a function with domain α with range cofinal in b, and such that $cf(\beta) > cf(\alpha)$. Choose your example so that $\alpha < \beta$ if you are able.

29. Prove that there exists a strong limit ordinal.

30. Let κ be an infinite cardinal. Prove that $cf(\kappa)$ is equal to the smallest cardinal λ such that $\kappa = \bigcup \mathcal{I}$ for some set $\mathcal{I}$ of cardinality λ such that for all $X \in \mathcal{I}$, $|X| < \kappa$.

31. Prove that the Axiom of Choice is equivalent to Tukey's Lemma (see Kunen, problem #11, on page 44).

32. (Corrected from Kunen, problem #14, p. 44) Prove $|\{X \subset \kappa : |X| = \lambda\}|$ is equal to κ^λ.

33. Prove that the Pairing Axiom follows from the rest of the axioms of ZF^-.

34. Which axioms of set theory are true in the $\in$-model with universe ω (and the standard $\in$ relation).

35. (Kunen, #19, p. 45) Suppose that if κ is an infinite cardinal and $\lhd$ is a well-ordering of κ. Prove there is a subset X of κ such that $|X| = \kappa$, and such that $\lhd$ and $<$ (i.e., $\lhd$ and $\in$) agree on X.

36.
a. Suppose κ is regular and $\lambda < \kappa$. Prove that κ^λ equals $\max\{\kappa, \sup\{\mu^\lambda : \mu < \kappa\}\}$.
b. Suppose κ is weakly inaccessible and $\lambda < \kappa$. Prove that κ^λ equals $\sup\{\mu^\lambda : \mu < \kappa\}$.
c. Suppose κ is inaccessible. Prove that $\kappa^{<\kappa} = \kappa$.

37. Suppose that κ is singular and is not a strong limit cardinal. Prove that $k^{<\kappa} = 2^{<\kappa} > \kappa$.

38. Suppose that κ is singular and is a strong limit cardinal. Prove that $2^{<\kappa} = \kappa$ and that $\kappa^{<\kappa} = \kappa^{cf(\kappa)} > \kappa$.

39. Prove that if M is a transitive model, then M satisfies Extensionality.

40. Prove that $\mathbf{HF} = V_\omega$.

41. What axioms of ZF are true in $V_{\omega+\omega}$?

42. Prove that the symmetry and transitivity of equality are provable in first-order logic. I.e., prove that

$$\vdash (\forall x)(\forall y)(x = y)$$
$$\vdash (\forall x)(\forall y)(\forall z)(x = y \wedge y = z \rightarrow x = z)$$

43. Let φ be any formula. Prove that $\vdash (\forall x)(\neg\varphi) \rightarrow \neg(\exists x)\varphi$, and that $\vdash (\exists y)[((\exists x)(\varphi(x))) \rightarrow \varphi(y)]$.

44. Fix an arbitrary language L.

a. Prove there is an infinite set T of sentences such that, for every L-structure $\mathcal{M}$, we have $\mathcal{M} \vDash L$ iff $|\mathcal{M}|$ is infinite.

b. Prove that there is no set T of sentences such that, for every L-structure $\mathcal{M}$, we have $\mathcal{M} \vDash L$ iff $|\mathcal{M}|$ is finite.

c. Prove that there is no finite set T of sentences such that, for every L-structure $\mathcal{M}$, we have $\mathcal{M} \vDash L$ iff $|\mathcal{M}|$ is infinite.

45. Suppose that $S \vDash T$ and $T \vDash S$ and that S is finite. Prove that there is a finite $R \subseteq T$ such that $R \vDash T$.

46. Let x, y be in $\mathbf{WF}$.

a. Prove that $\cup x$ and $\wp(x)$ are in $\mathbf{WF}$.

b. Prove that $\{x, y\}$, $x \times y$, ${}^y x$ are in $\mathbf{WF}$.

c. Compute the ranks of these set in terms of the ranks of x and y.

d. Compute the ranks of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.

47. Prove that $|V_\omega| = |\omega|$ without using the Axiom of Choice by giving an explicit isomorphism between the two sets. Hint: Given $n, m \in \omega$, let nEm hold iff the n-th bit of m's binary representation is equal to 1. Prove that $(V_\omega, \in)$ is isomorphic to (ω, E). (See problem III.5 in Kunen.)

48. Kunen, problem 15, page 108. Let AR be the strenthened Axiom of Replacement:

$$\forall x \in a \exists y \varphi \rightarrow \exists b \forall x \in a \exists y \in b \varphi.$$

Prove that AR is a theorem of ZF (with Foundation). See Kunen for a hint.

49. Kunen, problem 17, page 108. Prove in ZF, that if $\mathbf{R}$ is a well-founded relation on a class $\mathbf{A}$ (not necessarily set-like), then every non-empty subclass $\mathbf{X}$ of $\mathbf{A}$ has an $\mathbf{R}$-minimal element. See Kunen for a hint!

50. (Kunen, page 147, #4.) Show that if $\kappa > \omega$, then $|H(\kappa)| = 2^{<\kappa}$.

51. (Kunen, page 147, #5.) Prove, for all $\kappa > \omega$, that $H(\kappa) = V_\kappa$ iff $\kappa = \beth_\kappa$.

52 . Let κ be strongly inaccessible. Prove that "α is strongly inaccessible" is absolute for $H(\kappa)$.

54. Kunen, page 147. #7.

55. Kunen, page 147. #8. Further hint, see Theorem I.7.6 on page 17.

56. Prove that the composition of absolute functions is absolute.

57. Prove that (ZFC proves that)

$$Con(ZFC^-) \;\Rightarrow\; Con(ZFC^- + (\exists x)(x = \{x\})).$$

Kunen's problems 18 and 19 on page 148 have hints for this.

58. Prove also that the sentence φ in Gödel's incompleteness theorem can be picked so that both φ and $\neg\varphi$ unprovable. [Hint: look up Rosser's theorem.]

59. (Kunen, #2, page 180.) (AC) Let $\alpha \geq \omega$. Show that $|L(\alpha)| = |V_\alpha|$ if and only if $\alpha = \beth_\alpha$.

60. Kunen, #3, page 180. Assume $V = L$. Show that if $\alpha > \omega$, then $L(\alpha) = V_\alpha$ if and only if $\alpha = \beth_\alpha$.

61. Kunen, #4, page 180. Assume $V = L$. Show that $L(\kappa) = H(\kappa)$ for all infinite cardinals κ.

62. Kunen, #6, page 180. Definition and properties of $L(A)$.

63. Kunen, #7, page 180.