Math 267A - Set Theory - Spring 1995
Sam Buss
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Homework problems

1. Prove: $(\forall x)(x \notin x)$. Hint: Use Foundation and Comprehension.
2. Prove: $(\forall x, y, z)(\exists u)(x \in u \wedge y \in u \wedge z \in u)$. (The *Triple-Set Axiom*)
3. Prove that $\langle a, b\rangle = \langle c, d\rangle$ if and only if $a = c$ and $b = d$.
4. Prove that ω exists. (Use the definition of ω given in class, not the one in the text.)
5. Let AC' be the version of the Axiom of Choice that states that for every family $\mathcal{F}$ of non-empty, pairwise disjoint sets, there is a set Y such that for all $A \in \mathcal{F}$, $|A \cap Y| = 1$. Prove
   (a) $ZFC \vDash AC'$.
   (b) $ZF + AC' \vDash AC$.
6. Prove that the set $R(\omega)$ exists.
7. What axioms of ZFC are true in the structure $\langle R(1), \in\rangle$?
8. Prove that all axioms of $ZFC - Inf$ are true in $\langle R(\omega), \in\rangle$, and Inf is false in this structure. Conclude that $ZFC - Inf \nvDash Inf$. [You do not have to show that your proof is formalizable in ZFC.]
9. Prove that if x is transitive and totally-ordered by ϵ, then x is an ordinal.
10. Prove the following simple facts about ordinals.
    (a) $\alpha \subset \beta$ if and only if $\alpha \leq \beta$.
    (b) If α is a limit ordinal, then $\alpha = \cup\{\beta : \beta < \alpha\}$.
    (c) For any ordinal α, $\alpha = \cup\{S(\beta) : \beta < \alpha\}$.
    (d) For any ordinal, $\alpha < S(\alpha)$, and there is no ordinal β, such that $\alpha < \beta < S(\alpha)$.
11. Let X be a set of ordinals. Prove that $sup(X) = \bigcup X$. Here $sup(X)$ is defined to be the least ordinal greater than or equal to all ordinals in X.

12. Let X be a set of ordinals and define $\min(X) = \bigcap X$. Prove that $\min(X)$ is an ordinal, is in X and is less than or equal to all members of X.

13. Prove:

    (a) If $\beta \leq \beta'$, then $\alpha + \beta \leq \beta'$.

    (b) Let X be a set of ordinals. Then $\alpha + sup(X) = sup\{\alpha + \beta : \beta \in X\}$.

    (c) $\alpha \leq \beta$ iff $(\exists! \gamma)(\alpha + \gamma = \beta)$. [Suggestion: For the existence of γ, use induction on β.]

14. Prove $\alpha \in R(\alpha + 1) \setminus R(\alpha)$.

15. Prove that $\aleph_\beta$ is a limit cardinal if and only if β is a limit ordinal.

16. Does there exist a β such that $\aleph_\beta = \beta$?

17. Prove that the following are equivalent:

    (a) $rank(x) \leq \alpha$,

    (b) $\forall y \in x (rank(y) < \alpha)$.

    So $x \in R(\alpha) \iff x \subset \cup_{\beta \in a} R(\beta)$.

18. Prove $rank(x) = sup\{rank(y) + 1 : y \in x\}$.

19. Prove the following are equivalent:

    (a) x is hereditarily finite.

    (b) x is finite and all of its subsets are hereditarily finite.

    (c) x is in $R(\omega)$, i.e., $rank(x) < \omega$.

20. Suppose x and y are sets in $R(\alpha)$. Prove that $\{x, y\}$, $\cup x$, $x \times y$ and ${}^y x$ are in $R(\alpha + n)$ for some $n \in \omega$. (Try to make n as small as possible for each of these.)

21. Suppose that κ is strongly inaccessible. Prove that the axioms of ZF are true in $(R(\kappa), \in)$. Does your proof work if κ is only weakly inaccessible?

22. Let R^* be the following strengthening of the axiom of replacement:

$$(\forall A)[(\forall x \in A)(\exists y)\phi \rightarrow (\exists B)(\forall x \in A)(\exists y \in B)\phi].$$

    Prove that $ZF \vDash R^*$.

23. Prove that $ZF + \neg Con(ZF)$ does not have a transitive model (M, ϵ). [Hint: consider the number of symbols in a ZF proof of a contradiction.]

24. Prove that $ZF + Con(ZF)$ does not prove that there is a transitive model of ZF.

25. Prove that ZF + *1-consistency*(ZF) proves that ZF has a transitive model. (Well, this is conjectured by Sam to be true, but he does not have a proof at the moment. If any set theorists are surfing the net and can provide a proof, we would be grateful.)

26. Suppose $\phi(\vec{x})$ is a Σ_1 formula and $\psi(\vec{x})$ is a Π_1 formula. Also suppose that $\boldsymbol{M}$ is transitive and that $(\forall \vec{x})(\phi(\vec{x}) \leftrightarrow \psi(\vec{x})$ is true in both $\boldsymbol{M}$ and $\boldsymbol{V}$. Prove that ϕ and ψ are absolute for $\boldsymbol{M}$.

27. For κ an infinite cardinal, prove that $H(\kappa) = R(\kappa)$ if and only if $\kappa = \beth_\kappa$.