Math 261C: Randomized Algorithms

Lecture topic: Proof of PCP Theorem, Part II

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1. Preprocessing Step for PCP Theorem

From the previous lecture, we had a binary CSP and a graph $G = ((V, E), \Sigma, C)$. This G is such that that if G satisfies all constraints, then $UNSAT(G) = 0$ and if G does not satisfy all constraints, then $UNSAT(G) \geq 1/|G|$. We will do a series of transformations on G to widen the gap $[0, 1/|G|]$.

We make no assumptions on the degree of nodes of G. We want to get a graph G' which:

1. is an expander graph.
2. is d-regular.
3. has a self loop at each node.

We do this by using the following theorem.

Theorem 1. *There are constants $d \geq \lambda$ and $\beta > 0$ such that any constraint graph has an associated graph $G' = ((V', E')\Sigma, C')$ such that*

1. *G' is d-regular and has self loops*
2. $\lambda(G') \leq \lambda \leq d$
3. $\Sigma' = \Sigma$
4. $\beta \cdot UNSAT(G) \leq UNSAT(G') \leq UNSAT(G)$

Proof. The proof proceeds by building the graph G' in two steps.

Step 1:
Form $G_1 = prep_1(G)$ as follows.
Replace each vertex v of G by a small expander graph X_v of $deg(v)$ nodes which is d-regular and has algebraic expansion $\lambda_0 \leq d$ and combinatorial expansion h_0. Formally,

1. $V_1 = \{(v, e) : v \in V, e \in E\}$

2. $E_1 = \{\{(v_1, e), (v_2, e)\} : e = \{v_1, v_2\} \in E\} \cup \bigcup_{v \in V} E(X_v)$
3. $C(\{(v_1, e), (v_2, e)\}) = C(e)$ (same as in G)
4. $\underset{(u \in X_v)}{C(u)} = \{(a, a) : a \in \Sigma\}$ "Equality constraint"
5. Σ_1 is still Σ.
6. G_1 is $d + 1$-regular

Claim 1. $UNSAT(G) = 0 \implies UNSAT(G_1) = 0$

Proof. Assign the same symbol as node v to all nodes of the graph X_v □

Claim 2. *For some constant β_1, $UNSAT(G_1) \geq \beta_1 UNSAT(G)$*

Proof. Suppose σ_1 is an assignment to V_1. We want to construct an assignment σ to V such that $UNSAT_{\sigma_1}(G_1) \geq \beta_1 UNSAT_{\sigma}(G)$.
Let $\sigma(v) = a$ be such that $|\{e|\sigma_1(v, e) = a\}|$ is maximized.
Let $e = \{v_1, v_2\}$ be any edge for which the constraint in G is unsatisfied. There are two possible cases

Case 1: $\sigma_1(v_1, e) = \sigma(v_1)$, $\sigma_1(v_2, e) = \sigma(v_2)$
In this case, the edge $\{(v_1, e), (v_2, e)\}$ is unsatisfied.

Case 2: $\sigma_1(v_1, e) \neq \sigma(v_1)$, $\sigma_1(v_2, e) \neq \sigma(v_2)$
Consider the set $S_v^{a_1'}$ of vertices in X_{v_1} which get the label $a_1' = \sigma_1(v_1, e)$. Since a_1' is not the majority, we have $|S_v^{a_1'}| \leq |X_v|/2$. Using the expansion property, we have $|E(S_v^{a_1'}, \overline{S_v^{a_1'}})| \geq h_0 |S_v^{a_1'}|$. Since each edge in $E(S_v^{a_1'}, \overline{S_v^{a_1'}})$ violates the equality constraint, there is an unsatisfied edge within X_v. This allows us to allocate fraction $1/h_0$ of an unsatisfied edge in G to the unsatisfied constraint of G_1. Since each edge can participate at most twice in such an allocation (as this may happen also from the other endpoint of the edge), we get at least a fraction $1/(2h_0)$ of an unsatisfied edge constraint in G_1 for each unsatisfied edge in G.

Since $|E(X_v)| \leq d \cdot deg(v)$ we get $|E(G_1)| \leq |E(G)|(d + 1)$. Choosing $\beta_1 = 1/(2h_0(d + 1))$ completes the proof. □

We now move to the second transformation which converts it into an expander graph with self loops.
Form $G_2 = prep_2(G_2)$ and set $G' = G_2$.
We use the following construction.

Let Y be a d'-regular expander graph with expansion $\lambda' \leq d$ on the vertices of G'

1. $V(G_2) = V(G_1)$

2. $E(G_2) = E(G_1) \cup E(Y) \bigcup_{v \in V_1} \{v, v\}$

3. Constraints on G_2 are
   - Unchanged for edges of G_1
   - Null constraints for self loop and edges of Y

It is not hard to show that G_2 is $d + 1 + d' + 1$-regular and has expansion $\lambda(G_2) \leq d + 1 + \lambda_0 + 1 \leq d + 1 + d' + 1$.

Setting $G' = G_2$ completes the proof. $\square$

2. Amplification Construction

Given a graph $G = ((V, E), \Sigma, C)$, which is d-regular, we want to form the graph G^t. The graph G^t is formed as follows

1. Vertices: $V(G^t) = V(G)$
2. Edges: $E(G^t) =$ The multiset $\{\{v_1, v_2\} |$ there is a t-step walk in G from v_1 to $v_2\}$
3. Alphabet: The alphabet of G^t is $\Sigma^{d^{\lceil t/2 \rceil}}$. Given a value $\sigma(v) = a \in \Sigma^{d^{\lceil t/2 \rceil}}$, it can be interpreted as v's assignment from the alphabet Σ to the at most $d^{\lceil t/2 \rceil}$ many distinct nodes of G reachable from v in exactly $\lceil t/2 \rceil$ steps. The ordering of the nodes is chosen by fixing some canonical ordering.
4. Constraints: Let $\Gamma(u)$ be the set of vertices of G reachable from u in exactly $\lceil t/2 \rceil$ steps. Let v_1, v_2 be any pair of nodes in $\Gamma(u)$. Supppose the nodes v_1, v_2 be assigned values $a_1, a_2 \in \Sigma^{d^{\lceil t/2 \rceil}}$ respectively. Then for any edge in $E \cap \Gamma(v_1) \times \Gamma(v_2)$, the value assigned in a_1, a_2 must satisfy the original constraint in G. The constraint in G^t is satisfied iff this holds for all such pairs v_1, v_2.