Math 261C: Randomized Algorithms

Lecture topic: WalkSat, part II & Lovász Local Lemma, part I

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1. WalkSat, Part II

We now analyze the WalkSat algorithm described in the previous lecture for values of k at least 3.

Recall that the WalkSat algorithm can be compared to a Markov process on the states $\{0, 1, 2, \ldots, n\}$, where n is the number of variables in the given instance of k-SAT. These states represent the "distance" from the algorithm's current generated assignment to the nearest satisfying assignment. Let q_j be the probability of reaching state 0 in at most $m = 3n$ steps starting from the state j.

We will allow i "rightward" (away from 0) moves before ending up at state 0. This requires $j+2i$ moves, which is less than $3n$ since $i, j \in \{0, 1, 2, \ldots, n\}$. The probability of a rightward move is $\frac{k-1}{k}$ and the probability of a leftward move is $\frac{1}{k}$.

Therefore we have

$$\begin{aligned} q_j &\geq P\,[\text{ending up at 0 after at most } j + 2i \text{ moves}] \\ &\geq \binom{j+2i}{i} \left(\frac{1}{k}\right)^{j+i} \left(\frac{k-1}{k}\right)^{i}. \end{aligned}$$

Let $i = \left\lceil \frac{j}{k-2} \right\rceil$ so that $i \approx \alpha j$ for $\alpha = \frac{1}{k-2}$. Then $j + 2i \approx \beta j$ for $\beta = 1 + 2\alpha = \frac{k}{k-2}$.

Then by Stirling's Formula (which states $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$) we have

$$\begin{aligned} \binom{j+2i}{i} &\approx \binom{\beta j}{\alpha j} \\ &= \frac{(\beta j)!}{(\alpha j)!((\beta - \alpha)j)!} \\ &\approx \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\beta}{\alpha(\beta-\alpha)}} \frac{1}{\sqrt{j}} \left(\frac{\beta^\beta}{\alpha^\alpha(\beta-\alpha)^{\beta-\alpha}}\right)^j. \end{aligned}$$

Now $\beta - \alpha = \frac{k-1}{k-2}$. So, ignoring the constants out in front, we can say that the order of the binomial coefficient in question is

$$\binom{j+2i}{i} \approx \frac{1}{j^{\Theta(1)}} \left(\frac{k^{k/k-2}}{(k-1)^{(k-1)/(k-2)}} \right)^j .$$

Using this in our lower bound for q_j we find

$$q_j > \frac{1}{j^{\Theta(1)}} \left(\frac{1}{k-1} \right)^j .$$

Now recall that the algorithm for WalkSat has the form

Algorithm: WalkSat(Γ, k, n)

Loop: Choose at random a truth assignment for the variables.
    Loop (at most 3n times): Check if formula is satisfied.
        If not, flip a random literal in a random unsatisfied clause.

The probability p that the inner loop finds a satisfying assignment (assuming one exists) can be estimated by

$$\begin{aligned}
p &> \sum_{j=0}^{n} P\,[\text{we disagree from a satisfying assignment in } j \text{ values}]\, q_j \\
&\geq \sum_{j=0}^{n} \left(\frac{1}{2^n} \binom{n}{j} \right) \frac{1}{n^{\Theta(1)}} \left(\frac{1}{k-1} \right)^j \\
&= \frac{1}{n^{\Theta(1)}} \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{k-1} \right)^j \\
&= \frac{1}{n^{\Theta(1)}} \frac{1}{2^n} \left(1 + \frac{1}{k-1} \right)^n && \text{by the binomial theorem} \\
&= \frac{1}{n^{\Theta(1)}} \frac{1}{2^n} \left(\frac{k}{k-1} \right)^n \\
&= \frac{1}{n^{\Theta(1)}} \left(\frac{1}{2} \frac{k}{k-1} \right)^n \\
&> \frac{1}{2^n},
\end{aligned}$$

and so the expected number of iterations of the outer loop before it succeeds is approximately

$$n^{\Theta(1)} \left(2 \frac{k-1}{k} \right)^n < 2^n .$$

This is an improvement over the previous algorithm, but still runs in exponential time.

Below is a table for the expected runtime of WalkSat for various values of k.

k	Expected Runtime
3	$\left(\frac{4}{3}\right)^n$
4	$\left(\frac{3}{2}\right)^n$
5	$\left(\frac{8}{5}\right)^n$
k	$(s_k)^n$

Compare this to the conjectured runtimes for deterministic algorithms for SAT.

Exponential Time Hypothesis: Let $k \geq 3$. Any deterministic algorithm for SAT takes time $n^{\Theta(1)}(s_k)^n$ in k-SAT instances for some constant $s_k > 1$.

Strong Exponential Time Hypothesis: For any deterministic algorithm for SAT, $\lim_{k\to\infty} s_k = 2$.

For a derandomization of the WalkSat algorithm see [1].

2. Lovász Local Lemma, Part I

Definition. *Let $E_1, E_2, \ldots, E_n$ be events in some probability space. We say G is a dependency graph for $E_1, \ldots, E_n$ if G has vertex set $\{1, 2, \ldots, n\}$ and for all i E_i is independent of the set $\{E_j :$ there is no edge from i to $j\}$.*

We will use the notation

$$\Gamma_i = \{j : \text{ there is an edge from } i \text{ to } j\}.$$

Lovász Local Lemma. *Let $E_1, \ldots, E_n$ and G be as in the definition. Also let $x_1, x_2, \ldots, x_n \in [0, 1)$ be such that*

$$P[E_i] \leq x_i \prod_{j\in\Gamma_i} (1 - x_j).$$

Then

$$P[\overline{E_1} \wedge \ldots \wedge \overline{E_n}] \geq \prod_{i=1}^{n} (1 - x_i).$$

Corollary. *Suppose G has degree at most d (i.e., for all i we have $|\Gamma_i| \leq d$). Suppose also that $P[E_i] \leq p$ for all i. If either*

$$ep(d + 1) \leq 1 \text{ or } 4pd \leq 1$$

then

$$P(\overline{E_1} \wedge \ldots \wedge \overline{E_n}) > 0.$$

Proof of the Corollary. Let $x_i = \frac{1}{d+1}$ for all i. To apply the Lovász Local Lemma we need to show that

$$p \leq \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^d.$$

This will show that $P(E_i) \leq x_i \prod_{j \in \Gamma_i}(1 - x_j)$ and it will follow that $P(\overline{E_1} \wedge \ldots \wedge \overline{E_n}) > 0$.

In the first case, if $ep(d+1) \leq 1$, it suffices to show that $\frac{1}{e} \leq \left(1 - \frac{1}{d+1}\right)^d$. We can show this easily by taking the logarithm of both sides.

Otherwise, if $4pd \leq 1$, it suffices to show that

$$\frac{1}{4d} \leq \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^d = \frac{1}{d+1}\left(\frac{d}{d+1}\right)^d.$$

This is true if and only if

$$\frac{1}{4} \leq \left(\frac{d}{d+1}\right)^{d+1} = \left(1 - \frac{1}{d+1}\right)^{d+1}.$$

Taking logarithms, we need to verify that

$$-\log(4) \leq (d+1)\log\left(1 - \frac{1}{d+1}\right),$$

which is true for all $d \geq 1$. Thus the corollary is proved. $\square$

For an improved, and optimal, version of the Corollary, see [2].

References

[1] Moser, R.A. and D. Scheder. *A Full Derandomization of Schoening's k-SAT Algorithm.* SToC 2011.
[2] Shearer. *On a Problem of Spencer.* Combinatorica 5 (1985) 241-245.