Math 260AB - Introduction to Mathematical Logic

Winter and Spring 2008

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Homework I. Due: Monday, January 23.

General instructions for all homeworks: Proofs are not required unless the problem asks for a proof. Don't forget conventions on omitting parentheses.

The notation "**NA**" means "not assigned".

NA-1. Explain the Russell and Whitehead "dot notation" used in *Principia Mathematica*. (See page 10 of volume 1 of *Principia Mathematica* which is available online from the University of Michigan, or see the entry in the online Stanford Encyclopedia of Logic on the notation used in the *Principia Mathematica*.) Specifically: describe how to convert from parentheses to dot notation, and vice-versa.

1. Prove that $\{\land, \lor, \to\}$ is not a functionally complete set of connectives.

2. Prove that $\{\neg, \leftrightarrow\}$ is not a functionally complete set of connectives. (For the latter, see also problem 7 below.)

3. Let the Shaeffer stroke (NAND) be defined so that $p|q$ means the same as $\neg(p \land q)$. Prove that $\{|\}$ is functionally complete.

4. Prove that $\{\to, \bot\}$ is functionally complete.

5. Let $\downarrow$ be the NOR connective; $(p \downarrow q) \vDash\!\!\!\Dashv \neg(p \lor q)$. Prove that there is no other binary Boolean connectives besides $\downarrow$ and $|$ that is functionally complete. [Hint: there are 16 binary Boolean connectives, including constant functions and functions that depend on only one input.]

6. Use a truth table to prove that $(p \leftrightarrow q) \leftrightarrow r$ is tautologically equivalent to $p \leftrightarrow (q \leftrightarrow r)$.

7. Describe succinctly the conditions under which the formula

$$p_1 \leftrightarrow p_2 \leftrightarrow p_3 \leftrightarrow \cdots \leftrightarrow p_n$$

is true. That is, give a natural characterization of when a truth assignment assigns this formula the value True.

8. Let $p_{i,j}$ be distinct variables, for $1 \leq i \leq m$ and $1 \leq j \leq n$. Consider the CNF formula

$$\varphi \ := \ \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} p_{i,j}.$$

Describe the formula φ^{DNF} that is (a) equivalent to φ and (b) is in disjunctive normal form. (Say, as obtained from φ by using distributivity of $\land$ and $\lor$.) How large is the formula φ^{DNF}? (Suggestion: measure the size of φ^{DNF} by the number of occurrences of variables in the formula.)

Can you prove your bound on the size of the DNF formula is optimal?

9. Give an example of a set Γ of formulas which is not tautologically equivalent to any finite set Δ of formulas.

10. Suppose that Γ is tautologically equivalent to some finite set Δ. Prove that there is a finite subset Γ_0 of Γ such that Γ and Γ_0 are tautologically equivalent.

11. Suppose that Γ and Δ are sets of formulas that express the negation of each other. Namely, suppose that any given truth assignment τ satisfies exactly one of Γ or Δ. Prove that Γ (and similarly Δ) is tautologically equivalent to some finite set formulas.