Math 260B — Mathematical Logic — Scribe Notes
UCSD — Spring Quarter 2012
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1 Arithmetizing Metamathematics Continues

Substitution Operation

A particularly important syntactic operation is the substitution of a term into a formula. Let A be a formula $A(a_i)$, and t a (semi) term. Because of the sequent calculus' conventions on free and bound variables, one can always form the formula $A(t/a_i)$, which is A with t substituted in for the free variable a_i, by replacing each occurrence of a_0 as a subexpression of A with the expression t.

$Sub(u, v, w)$ denotes the function such that for all formulas A and terms t.

$$Sub(\ulcorner A \urcorner, \ulcorner a_i \urcorner, \ulcorner t \urcorner) = \ulcorner A(t/a_i) \urcorner$$

$I\Sigma_1$ can Σ_1-defines $Sub(u, v, w)$ and prove its simple properties. For example,

(i) $I\Sigma_1 \vdash (\forall w, u, v)[\mathit{Wff}(\ulcorner w \urcorner) \wedge \mathit{Term}(\ulcorner u \urcorner) \wedge \mathit{FreeVar}(\ulcorner v \urcorner) \rightarrow \mathit{Wff}(Sub(w, u, v))]$;

(ii) In particular, $I\Sigma_1 \vdash \mathit{Wff}(\ulcorner A \urcorner) \wedge \mathit{Term}(\ulcorner t \urcorner) \wedge \mathit{FreeVar}(\ulcorner a_i \urcorner) \rightarrow \mathit{Wff}(Sub(\ulcorner A \urcorner, \ulcorner a_i \urcorner, \ulcorner t \urcorner))$;

Other concepts that $I\Sigma_1$ can Δ_0 define

(i) $\mathit{LogicalAxiom}(\ulcorner S \urcorner)$ – S is a valid initial sequent.
Here, $\ulcorner A \longrightarrow A \urcorner$ is $\ulcorner A \urcorner * \langle \longrightarrow \rangle * \ulcorner A \urcorner$.

(ii) $\mathit{ValidInference}_1(\ulcorner S_1 \urcorner, \ulcorner S \urcorner)$ – $\ulcorner S_1 \urcorner$ and $\ulcorner S \urcorner$ are Gödel numbers of sequents and $\dfrac{S_1}{S}$ is a valid inference.

(iii) $\mathit{ValidInference}_2(\ulcorner S_1 \urcorner, \ulcorner S_2 \urcorner, \ulcorner S \urcorner)$ – $\dfrac{S_1 \quad S_2}{S}$ is a valid inference.

(iv) $\mathit{Proof}(w)$ – w is the Gödel number of a valid proof P – $\ulcorner P \urcorner$ codes a sequence $\ulcorner S_1 \urcorner * \langle , \rangle * \ulcorner S_2 \urcorner * \langle , \rangle * \cdots \ulcorner S_n \urcorner$ and each S_j is either a logical axiom or infered by a valid inference from one or two earlier sequents.

(v) $Prf(\ulcorner P \urcorner, \ulcorner A \urcorner)$ – P is a proof of the sequent $\longrightarrow A$

Finally, the set of theorems can be defined by

$$Thm(\ulcorner A \urcorner) \iff \exists w Prf(w, \ulcorner A \urcorner).$$

Note that this is not generally Δ_1-definable in $I\Sigma_1$, not a primitive recursive property!, not a recursive property!! But this is an r.e. property.

Provability in theorems

Let T be a theory. Assume that T is given by a set of axioms:

Definition 1. T is axiomatizable iff there exists a recursive set Γ of axiom for T.

Theorem 1. *If T is axiomatizable, then T has a primitive recursive set of axioms.*

Theorem 2. *If T has an r.e. set of axioms, then T is axiomatizable.*

Assume that T has a primitive recursive set of axioms. Then,

$$Axiom_T(\ulcorner S \urcorner)$$

means S is "$\longrightarrow$ A," where A is an axiom of T. $Axiom_T$ is primitive recursive and Δ_1-definable in $I\Sigma_1$. Similarly, $Proof_T$, Prf_T, Thm_T can be defined.

Henceforth, $T \supseteq I\Sigma_1$ ($T \supseteq Q$ works too). For instance, T can be PA or $I\Sigma_1$.

Our goal is to show that (in English)

$$I\Sigma_1 \vdash \text{“If } T \vdash A, \text{then there is a proof that } T \vdash A\text{”},$$

and (in symbols)

$$I\Sigma_1 \vdash (\forall \ulcorner A \urcorner)[Thm_T(\underline{\ulcorner A \urcorner}) \to Thm_{I\Sigma_1}(\underline{\ulcorner Thm_T(\underline{\ulcorner A \urcorner}) \urcorner}).\text{(Intentional Property)}$$

$$I\Sigma_1 \vdash (\forall w)[Wff(w) \wedge Thm_T(w) \to Thm_{I\Sigma_1}(Sub(\underline{Thm_T(a_0)}\urcorner, \underline{\ulcorner a_0 \urcorner}, Num(w)))].$$

Here are notices. "We have defined $\ulcorner A \urcorner$ in a nonconventional manner: the usual definition is to let $\ulcorner A \urcorner$ represent a closed term whose value is equal to the Gödel number of A. We shall represent this alternative concept with the notation $\underline{\ulcorner A \urcorner}$. The definition for $\ulcorner A \urcorner$ that we are using is better for our intensional development" (Chapter 2, page 115, footnote 10).

$Num(x)$ is $\ulcorner S(S(\cdots(0)))\urcorner$. This is Σ_1 definable in $I\Sigma_1$ and primitive recursive.

Take some fixed formula A. Let w be $\ulcorner A \urcorner$. $Wff(\underline{w})$ holds and is $I\Sigma_1$-provable. Then,

$$I\Sigma_1 \vdash [Thm_T(\underline{\ulcorner A \urcorner}) \rightarrow Thm_{I\Sigma_1}(\underline{\ulcorner Thm_T(\underline{\ulcorner A \urcorner}) \urcorner}).\text{(Extentional Property)}$$

$$I\Sigma_1 \vdash [Wff(\underline{w}) \wedge Thm_T(\underline{w}) \rightarrow Thm_{I\Sigma_1}(Sub(\underline{Thm_T(a_0)\urcorner}, \underline{\ulcorner a_0 \urcorner}, \underline{w}))].$$