Math 260A — Mathematical Logic — Scribe Notes
UCSD — Spring Quarter 2012
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Last time we talked about the Δ_0 definition of the graph of $2^x = y$.

Claim 1. *$I\Delta_0$ proves "simple" properties of exponentiation, such as:*

1. $(2^x = y) \to (2^{x+y} = 2y)$
2. $(2^x = y) \cup (2^{x'} = y') \to (2^{x+x'} = yy')$
3. $2^0 = 1$

Note that we technically proved only expressibility, not provability.
Recall how we incrementally expressed i and i^*:

i	$i*$
1	1
10	10
11	1
100	x100
101	x1

or

i	i^*
1	1
10	10
11	1
100	x100
101	1
110	x10

or

i	i^*
1	1
10	10
11	1
100	x100
101	1
110	x10
111	x1

or finally,

i	i^*
1	1
10	10
11	1
100	100
101	1
110	10
111	1
1000	x1000

Note that with each table, we add (in net) two more symbols from one table to the next. This can be formalized in $I\Delta_0$. Let

$$w = 1^*2^* \ldots k^*.$$

As we showed in class last week,

$$|w| = 2k - \text{Numones}(k).$$

Let $\overline{w}$ give w with the 'x's in the above table recorded as 1s.

Example 1. $w = 11011001$ and $\overline{w} = 1101110011$.

The sketched inductive argument from the tables above gives us that $|\overline{w}| = 2^k$.

Theorem 1. *$I\Delta_0$ can Δ_0 define* $\text{Numones}(x) = i \leftrightarrow (|\overline{w}| - |w| = 2i)$*, where* $|w| :=$ *least* $i \leq w : 2^i > w$

Example 2. $5 = (101)_2$ so $|5| = 3$ since $2^3 > 5 > 2^2$

Claim 2. *$I\Delta_0$ proves "simple" properties of* $\text{Numones}(x)$*, such as:*

1. $\text{Numones}(x) \leq |x|$
2. $\text{Numones}(2x) = \text{Numones}(x)$
3. $\text{Numones}(2x+1) = \text{Numones}(x) + 1$
4. *"z is a power of 2" implies* $\text{Numones}(z) = 1$
5. $\text{Numones}(x + 2^{|x|}y) = \text{Numones}(x) + \text{Numones}(y)$

1 An Alternated Method for Sequence Encoding Based on Numones

In base 4, use 2 as a comma. Thus represent $\langle a_0, \ldots, a_{k-1}\rangle$ as $2a_0 2a_1 \ldots 2a_{k-1}$, with each a_i written in binary notation. Then Numones can be used to count the number of 2s.

$w \to x; \forall i \leq x\, \text{Bit}(i,x) = 1 \leftrightarrow \text{Qit}(i,w) = 2\, \text{Bit}(2i{+}1, w) \cap \text{Qit}(i,w) = \text{Bit}(i,w){+}2$

Then $\text{Numones}(x) = \text{Numcommas}(w)$. Also let

$$\text{Len}(w) := \text{Numcommas}(w),$$

$$\text{Last}(w) := w \bmod 4^i, \text{ where } i \text{ is the least value such that } \text{Qit}(i,w) = 2,$$

$$\text{Pickout}(i,w) = \frac{w}{4^j} \text{ where } j \text{ least such that } \text{Len}(\frac{w}{4^j} = i),$$

$$\beta(i,w) := \text{Last}(\text{Pickout}(i+1, w))$$

Claim 3. *$I\Delta_0$ proves "simple" properties of sequence coding, along with Δ_0 defining $x \to \langle x\rangle$, $w \frown x$ (appending), and $w_1 * w_2$ (concatenating):*

1. $\beta(0, \langle x\rangle) = x$

   $\text{Len}(\langle x\rangle) = 1$

   $\text{Len}(\langle\rangle) = 0$

2. $\text{Len}(w \frown x) = \text{Len}(w) + 1$

3. $\mathrm{Len}(w_1 * w_2) = \mathrm{Len}(w_1) + \mathrm{Len}(w_2)$

4. $i < \mathrm{Len}(w) \to \beta(i, w) = \beta(i, w \frown x) = \beta(i, w * w)$

5. $\beta(\mathrm{Len}(w), w \frown x) = x$

   $\beta(i + \mathrm{Len}(w), w * w') = \beta(i, w)$

6. $|\langle a_0, \ldots a_{k-1}\rangle| = 2\left(k + \sum_{i=0}^{k-1} |a_i|\right)$, *where the sequence has no extraneous leading 0s, is in base 4 representation with no 3s, and any 2 in base four notation is not followed by a 0.*

Unassigned homework:

$$|\langle a_0, \ldots a_{k-1}\rangle| = \sum_{i=0}^{k-1} a_i$$

is Δ_0 definable in $I\Delta_0$.

Recall $I\Delta_0$ does not prove $\forall x \exists y (2^x = y)$.

Theorem 2. *$I\Sigma_1 \vdash \forall x \exists y (2^x = y)$.*

Proof. Recall that $I\Sigma_1$ is $I\Delta_0$ and induction on Σ_1 formulas. Arguing informally, $I\Sigma_1$ with define the sequence $w = \langle 1, 2, 4, 8, 16, \ldots$ where $\beta(i, w) = 2^i$. Then w satisfies $\phi(w)$ where

$$\phi(w) := (\beta(0, w) = 1) \cap (\forall i < \mathrm{Len}(w) - 1)\beta(i + 1, w) = 2\beta(i, w).$$

$$I\Sigma_1 \vdash \forall i \exists w (\phi(w) \cap \mathrm{Len}(w) = i + 1)$$

using induction on i.

$I\Sigma_1 \vdash \phi(0)$ using $w = \langle 1 \rangle$.

$I\Sigma_1 \vdash \phi(i) \to \phi(i + 1)$ using $w \to w \frown 2\beta(\mathrm{Len}(w) - 1, w)$.

Thus $I\Sigma_1 \vdash \forall i \phi(i)$

Then 2^x as a function is Σ_1 defined by

$$2^x = y \leftrightarrow \exists w (\mathrm{Len}(w) = x + 1 \cap \phi(w) \cap \beta(x, w) = y).$$

This Σ_1 definition of 2^x is $I\Sigma_1$. □

Theorem 3. *$I\Sigma_1$ can Σ_1 define every primitive recursive function.*

Proof. (Sketch.) By induction, starting with 0, S, and the projection maps, and then inductively working with composition and primitive recursion. (We show the latter.) Suppose $I\Sigma_1$ can Σ_1 define $g(x)$ and $h(m, s, x)$ and suppose

$$f(0, x) = g(x)$$

$$f(m + 1, x) = h(m, f(m, x), x).$$

We want to show that $I\Sigma_1$ can Σ_1 define f. By assumption we have

$$\Phi_y(\overline{x}, y) \leftrightarrow g(\overline{x}) = y,$$

$$\Phi_h(m, s, \overline{x}, y) \leftrightarrow h(m, s, \overline{x}) = y,$$

and that Φ_g and Φ_h are in Σ_1. Define $f(\overline{x}) = y$ by

$$w = \langle f(0, \overline{x}), dots, f(m, \overline{x})\rangle,$$

$$\begin{aligned} f(m, \overline{x}) =& y \leftrightarrow \exists w(\text{Len}(w) = m + 1 \cap \beta(0, w) = g(\overline{x}) \cap \\ &(\forall i < \text{Len}(w) - 1)(\beta(i + 1, w) = h(i + 1, \beta(i, w), x)) \cap \beta(m, w) = y). \end{aligned}$$

Thus

$$\begin{aligned} f(m, \overline{x}) =& y \leftrightarrow \exists w(\text{Len}(w) = m + 1 \cap \Phi_g(\overline{x}, \beta(0, w)) \cap \\ &(\forall i < \text{Len}(w) - 1)\Phi_h(i + 1, \beta(i, w), \overline{x}, \beta(i + 1, w))). \end{aligned}$$

□