Math 260A — Mathematical Logic — Scribe Notes
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1 Σ_1-defined Functions

Our goal has been to introduce Δ_0 defined relation symbols R and Σ_1-defined function symbols f to be added to $I\Delta_0$, and which are usable in induction. So we need Σ_1-defined functions now.

Recall that f is Σ_1-defined means that there is a Δ_0-formula φ (or a Σ_1-formula φ) such that:

- $T \supset I\Delta_0$
- $T \vdash \forall\vec{x}\exists!y(\varphi(\vec{x},y))$
- $T(f)$ is $T \cup \{\forall\vec{x}(f(\vec{(x)}) = y \leftrightarrow \varphi(\vec{x},y)\}$.

Note that $T(f)$ is conservative over T.

Claim. *For T a bounded theory, any $\Delta_0(f)$ formula is $T(f)$-provably-equivalent to a Δ_0 formula.*

First let's make a preliminary remark. By Parikh's Theorem, for $T \supset I\Delta_0$ and T bounded, if $T \vDash \forall\vec{x}\exists!y(\varphi(\vec{x},y))$, then there is a term q such that

$$T \vDash \forall\vec{x}\exists y \leq q(\vec{x})(\varphi(\vec{x},y)) \ .$$

And if φ is $\exists z_1 \ldots \exists z_k \psi(\vec{x},y,\vec{z})$ then

$$T \vDash \forall\vec{x}\exists y \leq q \exists z_1 \leq r_1 \ldots \exists z_k \leq r_k(\psi(\vec{x},y,\vec{z})) \ ,$$

again by Parikh's Theorem. So we have that Σ_1 definable implies Δ_0 definable.

Now back to the above claim. Let ψ be a $\Delta_0(f)$ formula. We want an equivalent ψ^* such that ψ^* is Δ_0 and $T(f) \vDash \psi \leftrightarrow \psi^*$. Here's our idea: We will remove occurrences of f one at a time.

So find an atomic formula containing an occurrence of f of form $s = t$ or $s \leq t$. That is,

$$s(\ldots f(r_1,\ldots,r_k)\ldots) = t$$

where there are no f's inside the terms r_i. This is equivalent to (in $T(f)$)

$$(\exists y \leq q(r_1, \dots r_k))(\varphi(\vec{x}, y) \wedge s(\dots y \dots) = t)$$

and this is clearly Δ_0.

Alternatively, we could consider

$$(\forall y \leq q(\vec{r}))(\varphi(\vec{x}, y) \rightarrow s(\dots y \dots) = t)$$

which is also Δ_0. In either case, we have shown the above claim.

Here's a more general theorem, which we will present without complete proof (since the proof is so similar to the discussion above).

Theorem 1. *If $f(\vec{(x)}) = y$ is Σ_1-defined and if $T \supset B\Sigma_1$, then any $\Sigma_i(f)$ formula (or $\Pi_i(f)$ formula) is $T(f)$ provably equivalent to a Σ_i (respectively Π_i) formula.*

The proof of this theorem works exactly in an analogous way as above, except without the bounds on the quantifiers.

2 Some Bootstrapping

We have already introduced some things with Δ_0 definitions:

- Restricted subtraction: $x \dot{-} y$
- x is prime
- x divides y: $x|y \leftrightarrow \exists z \leq y(x \cdot z = y)$

Claim. $I\Delta_0 \vdash (x \text{ is prime } \wedge x|a \cdot b \rightarrow x|a \vee x|b$.

But we'd like to have

$$I\Delta_0 \vdash \forall x(x \text{ has a unique prime factorization}) \ .$$

And in $I\Delta_0$ how can we even say that x has a unique prime factorization? We want something along the lines of

$$\forall \vec{x} \exists p_1, p_2, \dots p_k(\forall i, p_i \text{ is prime and } x = p_1 \cdots p_k) \ .$$

Or more specifically:

$$\forall \vec{x} \exists \langle p_1, \dots p_k \rangle (\forall i p_i \text{ is prime and } x = \Pi_i^k p_i) \ .$$

So we are almost there, we just need to talk a little bit about sequence coding. But here is a problem:

Theorem 2. *$I\Delta_0$ does not Δ_0-define the exponentiation function.*

Proof. Suppose it did. So

$$\varphi(x, y) \leftrightarrow x = 2^y$$

and

$$I\Delta_0 \vdash \forall y \exists x \varphi(x, y) \ .$$

Then by Parikh's Theorem,

$$I\Delta_0 \vdash \forall y \exists x \leq s(y) \varphi(x, y)$$

for some term s. And s is made up of 0, S, $+$, $\cdot$, so s is a polynomial.

So $s(y) \leq y^l + l$ for some $l \in \mathbb{N}$. So $\mathbb{N} \vDash \forall y \exists x \leq y^l + l(x = 2^y)$, which is clearly not true since $\forall y(2^y < y^l + l)$ is not true. □

And this is an issue for sequence coding since the Gödel method used prime powers.

Thus we have that any Δ_0 definable function of $I\Delta_0$ is bounded by a polynomial $s(y) \leq y^l + l$. So we have completely characterized the growth rates of function provably definable in $I\Delta_0$. (Note that this is what computer scientists call "linear growth rate functions.")

3 Some More Bootstrapping

Let's define some more things:

- *Predecessor*: $P(x) = x \dot{-} 1 = x \dot{-} S(0)$
- *Integer division*: $x, y \longmapsto \lfloor x/y \rfloor$ with $\varphi(x, y, z) \leftrightarrow (y \cdot z \leq x \wedge y \cdot Sz > x) \vee (y = 0 \wedge z = 0)$.
- $x \bmod y = x - y \cdot \lfloor x/y \rfloor$.
- $\lfloor \sqrt{x} \rfloor = z \leftrightarrow z \cdot z \leq x \wedge Sz \cdot Sz > x$.
- x is prime (we already did)
- x is a prime power $\leftrightarrow \exists p \leq x(p$ is prime and $\forall z(z|x \rightarrow z = 1 \vee p|z)$.
- x is a power of 2 (as in the previous bullet)
- (x is a power of 4)$\leftrightarrow$ (x is a power of 2 $\wedge$ ($x \bmod 3 = 1$)).