Math 260A — Mathematical Logic — Scribe Notes
UCSD — Winter Quarter 2012
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Wednesday, May 9, 2012

1 Remarks on Parikh's Theorem

We needed that T was a bounded theory. Also take $T \supset I\Delta_0$ (or at least $I(Open)$, induction on quantifier-free formulas) as an extra assumption.

We need the following assumtions.

$$T \vdash t_1 \leq t_1 + \cdots + t_k$$
$$T \vdash x \leq c^i \wedge y \leq c_j \to x \cdot y \leq c^{i+j}$$
$$T \vdash x \leq c^i \wedge y \leq c_j \to x + y \leq c^{i+j}$$
$$T \vdash x \leq c^i \to Sx \leq c^{i+1}$$

2 Finishing the Proof of Parikh's Theorem

Claim 2.1. *(Claim 3.) Let $T \supset Q + I(Open)$. Suppose $\mathcal{M} \models T$ and $\mathcal{M}'$ is an initial segment and a substructure of $\mathcal{M}$. Let $\psi = \psi(x_1, \ldots, x_k)$ be a Δ_0 formula, and let $m_1, \ldots, m_k \in \mathcal{M}'$. Then $\mathcal{M} \models \psi(m_1, \ldots, m_k) \iff \mathcal{M}' \models \psi(m_1, \ldots, m_k)$.*

Proof. Instead of contradiction, we'll do a proof by blahblahblah-tion, for some blahblahblah $\neq$ contradic. We'll use induction on the complexity of ψ.

The base case is that ψ is atomic; that is, ψ is $s = t$ or $s \leq t$ for some terms s, t on k variables. But the meanings of $+, \cdot, S, 0$ are the same in $\mathcal{M}'$ as in $\mathcal{M}$ by virtue of being a substructure, so the claim holds.

Now let $\psi = \neg\chi$. We have

$$\begin{aligned} \mathcal{M} \models \psi &\iff \mathcal{M} \not\models \chi \\ &\iff \mathcal{M}' \not\models \chi \\ &\iff \mathcal{M}' \models \psi \end{aligned}$$

by the induction hypothesis. The other induction steps for the logical connectors are clear.

Now suppose ψ is $(\exists y \leq s)\chi = \exists y(y \leq s \wedge \chi)$. Then

$$\begin{aligned} \mathcal{M} \models (\exists y \leq s(m_1, \ldots, m_k))\chi(y, m_1, \ldots, m_k) &\iff \exists m_0 \in |\mathcal{M}|(\mathcal{M} \models m_0 \leq s(m_1, \ldots, m_k) \\ &\qquad \text{and } \mathcal{M} \models \chi(m_0, m_1, \ldots, m_k)) \\ &\iff \exists m_0 \in |\mathcal{M}'|(\mathcal{M} \models m_0 \leq s(m_1, \ldots, m_k) \\ &\qquad \text{and } \mathcal{M} \models \chi(m_0, m_1, \ldots, m_k)) \\ &\iff \exists m_0 \in |\mathcal{M}'|(\mathcal{M}' \models m_0 \leq s(m_1, \ldots, m_k) \\ &\qquad \text{and } \mathcal{M}' \models \chi(m_0, m_1, \ldots, m_k)) \\ &\iff \mathcal{M}' \models (\exists y \leq s(m_1, \ldots, m_k))\chi(y, m_1, \ldots, m_k). \end{aligned}$$

The universal quantifier is now also clear, becase it's the negation of an existential quantifier. This completes the proof. □

Let R be Δ_0-defined by $T \supset I\Delta_0$. Then $T(R)$ is conservative over T, so that every $\Delta_0(R)$ formula is $T(R)$-provably equivalent to a Δ_0-formula.

Theorem 2.2. *(Theorem 1.) For T bounded (e.g. $I\Delta_0$), having R be Δ_1-defined suffices.*

Corollary 2.3. *In the above setting,* $T \vdash$ *induction for* $\Delta(R)$*-formulas.*

Proof. T proves Δ_0-induction. $\square$

So we can bootstrap by introducing an R which we can then use freely in the induction axioms (e.g. 's is prime').

Definition 2.4. Let $f : \mathbb{N}^k \to \mathbb{N}$ be a function. We say f is Σ_1 defined by $I\Delta_0$ if and only if it has a defining equation

$$\forall y \forall \vec{x} (f(\vec{x}) = y \longleftrightarrow \varphi(\vec{x}, y))$$

such that this is true in $\mathbb{N}$ and

$$I\Delta_0 \vdash \forall \vec{x} \exists ! y \varphi(\vec{x}, y)$$

for some $\varphi \in \Delta_0$ (sometimes we take $\varphi \in \Sigma_1$).

Remark 2.5. For $\varphi \in \Delta_0$, we get uniqueness 'almost' automatically. For suppose $I\Delta_0 \vdash \forall \vec{x} \exists y \varphi(\vec{x}, y)$. Let $\psi(\vec{x}, y)$ be $\varphi(\vec{x}, y) \wedge \forall y' < y \neg \varphi(\vec{x}, y')$. Then

$$I\Delta_0 \vdash \forall \vec{x} \exists ! y \psi(\vec{x}, y),$$

because $I\Delta_0 \vdash \Delta_0$-minimization.

Example 2.6. Define $f(x, y) = x \dot{-} y$ (restricted subtraction). Then

$$\varphi(x, y, z) := (x \geq y \wedge y + z = x) \vee (x < y \wedge z = 0).$$

The claim is now that $I\Delta_0 \vdash \forall x \forall y \exists ! z \varphi(x, y, z)$.

We argue in $I\Delta_0$. If $x \geq y$ then $y + z = x, y + z' = x$ implies that $z = z'$ by cancellation. Uniqueness in the other case is obvious. The existence of z is clear (by induction). So we win and f is defined by $I\Delta_0$.

Remark 2.7. Next time we'll show that Σ_1-defined functions can be added to $I\Delta_0$ and used freely in the induction axioms.