Math 260A — Mathematical Logic — Scribe Notes
UCSD — Spring Quarter 2012
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1 Quantifier Complexity and Bounded Quantifiers

So far we have used ordinary quantifiers $\forall$ and $\exists$. In order to study quantifier complexity, we now introduce bounded versions, defined here:

$$(\forall y \leq t)A(y) \leftrightarrow (\forall y)(y \leq t \rightarrow A(y))$$

$$(\exists y \leq t)A(y) \leftrightarrow (\exists y)(y \leq t \wedge A(y))$$

where t is a term not involving y.

Define a formula to be Δ_0 if all of its quantifiers are bounded.

We further define a sequence of classes of formulas.

- A Σ_1 formula has the form $(\exists y_1)\ldots(\exists y_k)\varphi(\vec{x},\vec{y})$, where φ is $Delta_0$.
- A Π_1 formula has the form $(\forall y_1)\ldots(\forall y_k)\varphi(\vec{x},\vec{y})$, where φ is $Delta_0$.
- A Σ_2 formula has the form $(\exists\vec{y})(\forall\vec{z})\varphi(\vec{x},\vec{y},\vec{z})$, where φ is $Delta_0$. Equivalently, it has the form $\exists\vec{y}\psi(\vec{x},\vec{y})$, where ψ is Π_1.
- Inductively, a formula is Σ_n if it has the form $\exists\vec{y}\varphi(\vec{x},\vec{y})$, where φ is Π_{n-1}.
- Π_n is defined dually.

Note In each of the above, we may take any of the quantifier blocks to be empty so that, for example, $\Sigma_n \subseteq \Sigma_{n+1}$.

We now consider restricted induction axioms. If Φ is a class of formulas (such as Δ_0 or Σ_3), the Φ induction axioms are

$$\{A(0) \rightarrow (\forall x)(A(x) \rightarrow A(Sx)) \rightarrow (\forall x)A(x) : A \in \Phi\}.$$

Denote by $I\Phi$ the axiom system $Q_{\leq} + \Phi$-induction axioms[1] For example, $I\Delta_0$ allows induction on all Δ_0 formulas. Last time we showed that $I\Delta_0$ proves $x + y = y + x$.

[1]It is possible to redefine the axioms of $Q_{\leq}$ to use only bounded quantifiers.

We define Peano Arithmetic $= PA = \bigcup I\Sigma_n = \bigcup I\Pi_n$.

More generally, we define classes Σ_n^+ and Π_n^+. Σ_2, for example, includes formulas of the form $(\forall u \leq t)\exists y \forall z \varphi(u, x, y, z)$, where φ is Δ_0. Simply put, you get a Σ_n^+ formula by taking any Σ_n formula and inserting bounded quantifiers wherever you like — including inside of a quantifier block.

Collection property / replacement property

The following is valid in $\mathbb{N}$:

$$(\forall y \leq t)(\exists z)\varphi(y, z) \rightarrow (\exists u)(\forall y \leq t)(\exists z \leq u)\varphi(y, z). \tag{1}$$

This serves to put a uniform bound on the z-values, which is possible since there are only finitely many y values being considered.

If φ is in Σ_n (for example), then 1 is called a Σ_n-replacement axiom.

Theorem Any Σ_n^+ formula is equivalent to a Σ_n formula, and any Π_n^+ formula is equivalent to a Π_n formula.

We prove the statement for Σ_n^+, and Π_n^+ follows dually.

Since the converse to 1 is trivial, we will instead show both directions. We work by induction on n.[2]

We take the inverse of the axiom, so we'll instead show:

$$(\exists y \leq t)\forall z \psi(y, z) \leftrightarrow \forall u \exists y \leq t \forall z \leq u \psi(y, z),$$

where $\psi = \neg\phi$.

Thus it is enough to show that, if $\chi \in \Sigma_n$, then so are $(\forall y \leq t)\chi$ and $(\exists y \leq t)\chi$.

Since $\chi \in \Sigma_n$, it has the form $\exists z_1 \ldots \exists z_k \psi(y, \vec{z})$. Thus we have

$$\begin{aligned}
(\forall y \leq t) \quad &\leftrightarrow \quad (\forall y \leq t)\exists z_1 \ldots \exists z_k \psi(y, \vec{z}) \\
&\leftrightarrow \quad (\exists u)(\forall y \leq t)(\exists z_1 \leq u)\exists z_2 \ldots \exists z_k \psi(y, \vec{z}) \\
&\leftrightarrow \quad (\exists u)(\forall y \leq t)\exists z_2 \ldots \exists z_k(\exists z_1 \leq u)\psi(y, \vec{z}). \\
&\leftrightarrow \quad [repeat\text{k-1}times] \\
&\leftrightarrow \quad (\exists u')(\forall y \leq t)(\exists z_1 \leq u') \ldots (\exists z_k \leq u')\psi(y, \vec{z}).
\end{aligned}$$

Note that the last portion of this formula, $(\exists z_1 \leq u') \ldots (\exists z_k \leq u')\psi(y, \vec{z})$, is a Π_{n-1}^+ formula, so the induction hypothesis gives us an equivalent Π_{n-1} formula $\psi'(y, \vec{u})$. This formula can now absorb the $(\forall y \leq t)$. Adding on the $(\exists u')$ on the front leaves us with a Σ_n formula, as desired.

[2] You may find yourself asking: "Are we allowed to do induction here?" Remember: we are doing induction ourselves, not in a restricted proof system.

Extending languages

We often use $I\Delta_0$ as our base theory. This can prove statements like $x+y = y+x$, which we proved earlier using quantifier-free induction.

Our language is only $\{0, S, +, \cdot, \leq\}$, but we will also want function symbols. We extend by conservative definitions, for example

$$z|x \leftrightarrow \exists u(z \cdot u = x)$$

and

$$Prime(x) \leftrightarrow x \neq 1 \wedge (\forall z)(z|x \rightarrow z = 1 \vee z = x).$$

These definitions are fine, but we should be mindful of unbounded quantifiers. In both cases, we can (and should) replace them with bounded quantifiers — both u and z may be bounded by x without changing the meaning.

Definition A predicate $R(x_1, \ldots, x_k) \subseteq \mathbb{N}^k$ is Δ_0 if there is a Δ_0 formula $\phi(\vec{x})$ so that

$$\mathbb{N} \vDash \forall \vec{x}(R(\vec{x}) \leftrightarrow \phi(\vec{x})).$$

Definition Let T be a theory. Let R be as above. Then $T(R)$ is the theory T in the language of T plus symbol R, whose axioms are the axioms of T, along with the axiom $Defn_R := \forall \vec{x}(R(\vec{x}) \leftrightarrow \phi(\vec{x}))$.

Theorem
(a) $T(R)$ is a conservative extension of T.
(b) Let T be a theory from $I\Delta_0, I\Sigma_n, I\Pi_n$. Any bounded (ie Δ_0) formula ψ of $T(R)$ is $T(R)$-provably equivalent to a bounded formula χ in the language of T.

Proof
We showed (a) last quarter.
For (b), find χ by replacing each instance of $R(\vec{t})$ in ψ by $\phi(\vec{t})$. This maintains the quantifier complexity, since ϕ is Δ_0.

Note: if ψ is Σ_n, so is χ, independent of the theory we're working in.