Math 260A — Mathematical Logic — Scribe Notes
UCSD — Spring Quarter 2012
Instructor: Sam Buss

Notes by: David Lorant [1]
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I will be denoting $S(x)$, the successor of x as Sx

Last Time

Recall in the last lecture we defined the theory Q:

- Usual FO symbols (including propositional connectives, quantifiers, equality)
- Non-logical symbols (0, S, $+$, $\cdot$)
- Axioms:
  1. $\forall x\,(Sx \neq 0)$
  2. $\forall x \forall y\,((Sx = Sy) \rightarrow (x = y))$
  3. $\forall x\,(x \neq 0 \rightarrow \exists y\,(Sy = x))$
  4. $\forall x\,(x + 0 = x)$
  5. $\forall x \forall y\,((x + Sy) = S(x + y))$
  6. $\forall x\,(x \cdot 0 = 0)$
  7. $\forall x \forall y\,((x \cdot Sy) = (x \cdot y) + x)$

Definition 1. $Q_{\leq}$: The conservative extension of Q that includes the inequality symbol $\leq$ by adding the axiom $x \leq y \leftrightarrow \exists z\,(x + z = y)$

Definition 2. A theory is said to be *bounded* if it is axiomatizable with a set of bounded formulas. We want to be able to treat bounded quantifiers separately from regular quantifiers.

$Q, Q_{\leq}$ are induction-free fragments of arithmetic. The axioms of $Q, Q_{\leq}$ do not imply many elementary facts about addition and multiplication, such as commutativity and associativity. We want a language stronger than Q.

[1]Based on handwritten class notes by Tanya Hall

1 Induction Axioms

Let $A(x)$ be a formula. Induction axiom for A is

$$A(0) \wedge (\forall x (A(x) \to A(Sx)) \to \forall x A(x))$$

$A(x)$ can have other free variables (parameters). The axiom for $A(x, \vec{y})$ is

$$A(0, \vec{y}) \wedge (\forall x (A(x, \vec{y}) \to A(Sx, \vec{y})) \to \forall x A(x, \vec{y}))$$

Definition 3. The theory of *Peano Arithmetic*, PA, is the theory $Q_{\leq}$ plus induction for all first-order formulas.

2 Minimization Axioms

The following are two equivalent statements of the minimization axioms:

$$\exists x A(x) \to \exists x (A(x) \wedge \forall y (y < x \to \neg A(y)))$$

$$\exists x A(x) \to \exists x (A(x) \wedge \neg \exists y (y < x \wedge A(y)))$$

Note that while $<$ is not technically in the language, we can use $y < x$ to abbreviate $y \leq x \wedge y \neq x$

The Minimization Axioms are often used as an equivalence to Complete Induction.

3 Complete Induction

$$\forall x [\forall y (y < x \to B(y)) \to B(x)] \to \forall x B(x)$$

If we take B to be $\neg A$ and push negations, then this is equivalent to minimization on $\neg B$. We will now show that induction on $\neg A$ is equivalent to minimization on A

$\forall x [\forall y (y < x \to \neg A(y)) \to \neg A(x)] \to \forall x \neg A(x)$
$\neg \forall x \neg A(x) \to \neg \forall x [\forall y (y < x \to \neg A(y)) \to \neg A(x)]$
$\exists x A(x) \to \exists x \neg [\forall y (y < x \to \neg A(y)) \to \neg A(x)]$
$\exists x A(x) \to \exists x \neg [A(x) \to \neg \forall y (y < x \to \neg A(y))]$
$\exists x A(x) \to \exists x \neg [\neg A(x) \vee \neg \forall y (y < x \to \neg A(y))]$
$\exists x A(x) \to \exists x [A(x) \wedge \forall y (y < x \to \neg A(y))]$

On the face of it, complete induction is weaker than ordinary induction because you have to assume more; the antecedent is stronger.

4 Power of Induction

What is induction good for? Unlike in $Q/Q_{\leq}$, with induction we get basic facts about addition and multiplication. For example, PA implies commutativity of addition:

Claim 1. $PA \vdash \forall x \forall y\,(x + y = y + x)$

Proof. by induction on x. Let $A\,(x, y)$ be $x + y = y + x$.
We will use the induction axiom $\underbrace{A\,(0, y)}_{(1)} \wedge (\forall x\,\underbrace{(A\,(x, y) \to A\,(Sx, y))}_{(2)} \to \forall x A\,(x, y))$

So, we need to show
(1) $PA \vdash 0 + y = y + 0$
(2) $PA \vdash (x + y = y + x) \to (Sx + y = y + Sx)$

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Since $y + 0 = y$ from an axiom, it is sufficient to show $PA \vdash 0 + y = y$

(1*) $PA \vdash 0 + y = y$

Proof. by induction on y. Let $B\,(y)$ be $0 + y = 0$.
Using Induction Axiom $\underbrace{B\,(0)}_{(a)} \wedge (\forall y\,\underbrace{(B\,(y) \to B\,(Sy))}_{(b)} \to \forall y B\,(y))$
We need to show
$(a)\, PA \vdash 0 + 0 = 0$
$(b)\, PA \vdash (0 + y = y) \to (0 + Sy = Sy)$

(a) $PA \vdash 0 + 0 = 0$

Proof. $0 + 0 = 0$ (by axiom) □

(b) $PA \vdash (0 + y = y) \to (0 + Sy = Sy)$

Proof.

1.	$0 + y = y$	(by hypothesis)
2.	$0 + Sy = S\,(0 + y)$	(by axiom)
3.	$0 + Sy = Sy$	(by 1,2)

□

Thus from $(a), (b)$ with induction $PA \vdash 0 + y = y$ (concluding 1*)

Thus, $PA \vdash 0 + y = y + 0$ (concluding 1)

(2) $PA \vdash (x + y = y + x) \to (Sx + y = y + Sx)$
Since $y + Sy = S(y + x)$ from an axiom, it is sufficient to show $PA \vdash Sx + y = S(x + y)$

(2*) $PA \vdash Sx + y = S(x + y)$

Proof. by induction on y. Let $C(x, y)$ be $Sx + y = S(x + y)$.
Using Induction Axiom $\underbrace{C(x, 0)}_{(c)} \wedge (\forall y \underbrace{(C(x, y) \to C(x, Sy))}_{(d)} \to \forall y C(x, y))$
We need to show
$(c)\, PA \vdash Sx + 0 = S(x + 0)$
$(d)\, PA \vdash (Sx + y = S(x + y)) \to (Sx + Sy = S(x + Sy))$

(c) $PA \vdash Sx + 0 = S(x + 0)$

Proof.

1. $Sx + 0 = Sx$ (by axiom)
2. $x + 0 = x$ (by axiom)
3. $Sx + 0 = S(x + 0)$ (by 1,2)

□

(d) $PA \vdash (Sx + y = S(x + y)) \to (Sx + Sy = S(x + Sy))$

Proof.

1. $Sx + y = S(x + y)$ (by hypothesis)
2. $x + Sy = S(x + y)$ (by axiom)
3. $S(x + Sy) = S(S(x + y))$ (by axiom)
4. $Sx + Sy = S(Sx + y)$ (by axiom)
5. $Sx + Sy = S(S(x + y))$ (by 1, 4)
6. $Sx + Sy = S(x + Sy)$ (by 3, 5)

□

Thus from $(c), (d)$ and induction axiom $PA \vdash Sx + y = S(x + y)$ (concluding 2*)

Proof.

1. $y + Sx = S(y + x)$ (by axiom)
2. $x + y = y + x$ (by hypothesis)
3. $y + Sx = S(x + y)$ (by 1, 2)
4. $Sx + y = y + Sx$ (by 2*, 3)

Thus $PA \vdash (x + y = y + x) \to (Sx + y = y + Sx)$ (concluding 2) □

Thus from (1) and (2) and induction, $PA \vdash (x + y = y + x)$ ■

5 Some things PA can prove

a) Addition is commutative: $\forall x \forall y \, (x + y = y + x)$

b) Addition is associative: $\forall x \forall y \forall z \, ((x + y) + z = x + (y + z))$

c) Multiplication is commutative: $\forall x \forall y \, (x \cdot y = y \cdot x)$

d) Distributive law: $\forall x \forall y \forall z \, ((x + y) \cdot z = x \cdot z + y \cdot z)$

e) Multiplication is associative: $\forall x \forall y \forall z \, ((x \cdot y) \cdot z = x \cdot (y \cdot z))$

f) Cancellation laws for addition: $\forall x \forall y \forall z \, (x + z = y + z \leftrightarrow x = z)$ and $\forall x \forall y \forall z \, (x + z \leq y + z \leftrightarrow x \leq z)$

g) Discreteness of $\leq$: $\forall x \forall y \, (x \leq Sy \to x \leq y \lor x = Sy)$

h) Transitivity of $\leq$: $\forall x \forall y \forall z \, (x \leq y \land y \leq z \to x \leq z)$

i) Anti-idempotency laws: $\forall x \forall y \, (x + y = 0 \to x = 0 \land y = 0)$ and $\forall x \forall y \, (x \cdot y = 0 \to x = 0 \lor y = 0)$

j) Reflexivity, trichotomy and antisymmetry of $\leq$: $\forall x \, (x \leq x) \, , \forall x \forall y \, (x \leq y \lor y \leq x) \, ,$ $\forall x \forall y \forall z \, (x \leq y \land y \leq x \to x = y)$

k) Cancellation laws for multiplication:$\forall x \forall y \forall z \, (z \neq 0 \land x \cdot z = y \cdot z \to x = y)$ and $\forall x \forall y \forall z \, (z \neq 0 \land x \cdot z \leq y \cdot z \to x \leq y)$

6 Prove $Q \vdash \forall x \neg \, (x < 0)$

Proof. Suppose $x < 0$. This means $x \leq 0 \land x \neq 0$. $x \leq 0$ means $\exists z \, (x + z = 0)$. By Q axiom, either $z = 0$ or $\exists z'$ such that $Sz' = z$.
If $z = 0$, then $0 = x + z = x + 0 = x$ which contradicts the fact $x \neq 0$
If $z = Sz'$, then $0 = x + Sz' = S \, (x + z)$ which contradicts the axiom $\forall x \, (Sx \neq 0)$ □

7 Complete Induction Axioms redux

We will now show $PA \vdash \forall x \left[\forall y \left(y < x \rightarrow A\left(y\right)\right) \rightarrow A\left(x\right)\right] \rightarrow \forall x A\left(x\right)$
We are going to use induction on the statement $B\left(x\right) = \forall y \left(y < x \rightarrow A\left(y\right)\right)$

Proof. Assume the hypothesis $\forall x \left[\forall y \left(y < x \rightarrow A\left(y\right)\right) \rightarrow A\left(x\right)\right]$.
Base Case $B\left(0\right)$ is $\forall y \left(y < 0 \rightarrow A\left(y\right)\right)$; so $Q \vdash B\left(0\right)$
Induction Step Assume $B\left(x\right)$. We want to show $B\left(Sx\right)$
So we assume $\forall y \left(y < x \rightarrow B\left(y\right)\right)$, and want to prove that $\left(y < Sx \rightarrow B\left(y\right)\right)$.
Assume $y < Sx$. By discreteness, we know $y \leq x$. This in turn means $y < x \vee y = x$.
If $y < x$, then $B\left(y\right)$ holds by our inductive hypothesis that $\forall y \left(y < x \rightarrow B\left(y\right)\right)$.
If $y = x$, then $B\left(y\right)$ holds from the hypothesis $\forall x \left[\forall y \left(y < x \rightarrow A\left(y\right)\right) \rightarrow A\left(x\right)\right]$
Thus by induction, $PA \vdash \forall y B\left(y\right)$. In particular, let x be arbitrary $B\left(Sx\right)$.,
so since $x < Sx$, $A\left(x\right)$ holds. ☐

8 $1 + 1 = 2$

Proof. Define $1 := S0$, and define $2 := SS0$
$S0 + S0 = S\left(S0 + 0\right) = SS\left(0 + 0\right) = SS0$ ☐