Math 260A — Mathematical Logic — Scribe Notes
UCSD — Winter Quarter 2012
Instructor: Sam Buss

Notes by: David Lorant [1]
Wednesday, March 07, 2012

1 Extensions by Definitions - Function Symbols

Recall in the last lecture we showed that we could take a sentence $\varphi(\vec{x})$ and add a relation symbol $P(\vec{x})$ to create a conservative extension. Today will will be doing a similar idea but with adding functions. The general idea is you take a formula $\varphi(x_1, \ldots, x_k, y)$ and define a graph of f: $y = f(x_1, \ldots x_k)$.

Theorem 1. *Let T_1 be a set of sentences in language L. Suppose $T_1 \models \forall\vec{x}\exists y\varphi(\vec{x}, y)$. Let L' be $L \cup \{f\}$, with f a new k-ary function. Let $T_2 = T_1 \cup \{\forall\vec{x}\varphi(\vec{x}, f(\vec{x}))\}$. Then T_2 is conservative over T_1.*

Proof. By Contradiction

Suppose A is an L-sentence, such that $T_2 \models A$ and $T_1 \not\models A$. Then there's a model $\mathcal{M} \models T_1, \mathcal{M} \not\models A$. Form $\mathcal{M}'$ which is an expansion of $\mathcal{M}$ to L', with $f^{\mathcal{M}'}(m_1, \ldots, m_k) =$ some m such that $\mathcal{M} \models \varphi[m_1, \ldots m_k, m]$. We can do this since $\mathcal{M} \models \forall\vec{x}\exists y\varphi(\vec{x}, y)$. Now $\mathcal{M}' \models T_2$ and $\mathcal{M}' \not\models A$, which is a contradiction.$\Rightarrow\Leftarrow$ □

Before, we had the following statement: Any L' formula A, has a corresponding L-formula A^* that means the same thing, but is in the smaller language via the translation $p(s_1, \ldots, s_k) \overset{*}{\longmapsto} \varphi(s_1, \ldots, s_k)$.

Unfortionately this doesn't work so well in this case. Suppose $T_1 \models \forall\vec{x}\exists y\varphi(\vec{x}, y)$ and we have a formula $A(f(s_1, \ldots, s_k))$ in the larger language. How do we eliminate the f?

One idea might be to take a new variable, say z and replace $A(f(s_1, \ldots, s_k))$ with $\exists z(\varphi(s_1 \ldots, s_k, z) \wedge A(z))$. This idea almost works, but requires one extra condition.

$$t_1 \vdash \forall\vec{x}\exists! yA(\vec{x}, y)$$

Definition 1. $\exists!$ (exists unique): $(\exists! y\varphi \equiv \exists y\varphi \wedge \forall y_1, y_2(\varphi(y_1) \wedge \varphi(y_2) \rightarrow y_1 = y_2))$

[1] Based on handwritten class notes by Matt Pead

If we have the condition $t_1 \vdash \forall \vec{x} \exists ! y A(\vec{x}, y)$, then we can say $\exists z (\varphi(s_1 \ldots, s_k, z) \wedge A(z))$. This would also be equivilent to $\forall z (\varphi(s_1 \ldots, s_k, z) \rightarrow A(z))$

For this result, we needed:

1. Unique existence condition
2. $=$ in the language
3. A quantifier free [2]

Note:

Assigning constant to z in $\exists z (\varphi(\vec{s}, z) \wedge A(z))$ doesn't help. This would give you $\exists z (\varphi(\vec{s}, z) \wedge (z = c) \wedge A(z))$, which would be equivalent to $(\varphi(\vec{s}, c) \wedge A(c))$. The trouble is there may be lots of s's occuring in the proof, or there may be quantified variables among the s's. So you can't just get one c that works.

Note:

Remember that we start off with the hypothosis of $T_1 \models A$, and end up with the conclusion that $T_2 \models A$. With the star translation it's easy to translate, but without it might be double exponential blowup of proof size? Maybe superexponential? (possible open question)

2 Skolemnization Process

Let φ be of the form $\forall \vec{x} \exists y [\psi(\vec{x}, y)]$, where no other variables appear free in ψ

Definition 2. Pre-Skolem form of φ is $\forall \vec{x} \exists y [\psi(\vec{x}, f(\vec{x}))]$ where f is a new function symbol. Denote this $\varphi^{(s')}$. Note that $\varphi^{(s')} \models \varphi$.

Definition 3. The Skolem Definition for φ is the formula $\forall \vec{x} [\exists y \psi(\vec{x}, y) \rightarrow \psi(\vec{x}, f(\vec{x}))]$

Claim 1. $\varphi +$ (*The Skolem definition for* φ) *is conservative over* φ*.*

Proof. $\varphi \models \forall \vec{x} \exists z [\exists y \psi(\vec{x}, y) \rightarrow \psi(\vec{x}, z)]$ since if there exists a y then z works, and if there isn't any y it doesn't matter what z you pick. □

Claim 2. *Let A be any sentence. Then* $\varphi \models A \Leftrightarrow \varphi^{(s')} \models A$*.*

[2] This is not a problem since we could apply the translation to atomic or quantifier free subformulas of A

Proof. $\Rightarrow$ Trivial since φ is weaker

$\Leftarrow$ Assume $\varphi^{(s')} \models A$. It follows that $\varphi \wedge$ (The Skolem definition for $\varphi) \models A$, since $\varphi \wedge$ (The Skolem definition for $\varphi) \models \varphi^{(s')}$. As shown above, $\varphi +$ (The Skolem definition for φ) is conservative over φ, so $\varphi \models A$ ☐

3 Why is the Skolemnization process useful?

Suppose φ is in prenex form.

Definition 4. Define φ^s to be $\left(\ldots\left(\left(\varphi^{s'}\right)^{s'}\right)^{s'}\ldots\right)^{s'}$ getting rid of all the existential quantifiers. That meas φ^s is universal in a language that is larger than φ, and φ^s is conservative over φ.

Definition 5. Let Γ be a set of sentences in prenex form. Define $\Gamma^s = \{\varphi^s : \varphi \in \Gamma\}$. Γ^s is conservative over Γ by the compactness theorem.

Remember that $\Gamma \models A$ is the same as "is $\Gamma \cup \{\neg A\}$ inconsistent?", which is the same as "is $(\Gamma \cup \{\neg A\})^s$ inconsistent?".

The next step will be to get rid of the universal quantifiers to make it equivalent to asking "is $\{A_1, \ldots, A_k\}$ inconsistent?", which in turn is the same as "is $B = (A_1 \wedge \cdots \wedge A_k)$ inconsistent?", which finally is the same as $\models \neg B$?

Since B is universal WLOG, $\neg B = \exists y_1, \ldots, \exists y_k C\left(x_1, \ldots, x_i, y_1, \ldots, y_k\right)$

Herbrand's Thorem states that $\models \exists y_1, \ldots, \exists y_k C\left(x_1, \ldots, x_i, y_1, \ldots, y_k\right) \Leftrightarrow$ $\exists$ a finite list of terms $t_{1,1}, \ldots t_{1,k}, t_{2,1}, \ldots t_{2,k}, \ldots, t_{m,1}, \ldots t_{m,k}$ such that $\models \bigvee_{j=1}^{s} C\left(\vec{x}, t_{j,1}, \ldots, t_{j,k}\right)$ (note this has no quantifiers). We have now reduced to quantifier free formulas, so now this is true iff it follows tautologically from equality axioms.