Math 260A — Mathematical Logic — Scribe Notes
UCSD — Winter Quarter 2012
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1 (Failed) Proof: Completeness for Uncountable Languages

Let L be a language of cardinality κ
Let Γ be a set of L-sentences.

Either there is a model M of Γ or there exists a finite $\Gamma_0 \subseteq \Gamma$ such that

$\Gamma_0 \to$

has a proof.

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Cardinality of Γ is $\leq \kappa$ (since the set of L-sentences has cardinality $=\kappa$)

Enumerate Γ as a well-ordered sequence $\gamma_0, \gamma_1, \gamma_2, ..., \gamma_\kappa$

Assume there is an inexhaustible supply of variables (need κ many)

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Form Λ, Ξ as before

$\Lambda \subseteq \Gamma$

Λ, Ξ satisfy closure properties (1)-(6)

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Do it in steps:

Definition:

$\Lambda_\alpha, \Xi_\alpha$ $\alpha \leq \kappa$

$\Lambda_{-1} = \Xi_{-1} = \emptyset$

Assume Λ_β, Ξ_β have been defined for all $\beta < \alpha$

Construct $\Lambda_\alpha, \Xi_\alpha$ such that there is no finite $\Gamma_0 \subseteq \Lambda_\alpha \cup \Gamma$, $\Delta_0 \subseteq \Xi_\alpha$ such that $\Gamma_0 \to \Delta_0$ has a proof, and such that $\Lambda_\alpha, \Xi_\alpha$ satisfy conditions (1)-(6), and $\gamma_\beta \in \Lambda_\alpha$ for $\beta \leq \alpha$.

To define $\Lambda_\alpha, \Xi_\alpha$ work backwards from the sequent

$\gamma_\alpha \to$

and try to give a proof.

Define "active":

$\Gamma' \to \Delta'$ is active if there is no finite $\Gamma_0 \subseteq \bigcup_{\beta<\alpha} \Lambda_\beta \cup\ \Gamma$, $\Delta_0 \subseteq \bigcup_{\beta<\alpha} \Xi_\beta$,
such that $\Gamma_0, \Gamma' \to \Delta_0 \Delta'$ has a proof.

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Form an "unproof" of $\gamma_\alpha \to$ as before, which has an infinite branch of nonactive sequents. Let:

$\Lambda_\alpha = (\bigcup_{\beta<\alpha} \Lambda_\beta) \cup \{$ formulas in antecedents of this finite branch$\}$, and
$\Xi_\alpha = (\bigcup_{\beta<\alpha} \Xi_\beta) \cup\ \{$ formulas in succedents of this finite branch$\}$.

Modifications:

Omit step 1.

Enumerate formulas that appear in the active sequent.

-Say, pick the last to have been worked with and it has $\exists x\varphi(x)$.

Problem: The following condition should hold, but does not:

If $\forall x\varphi(x)$ is in Λ, then $\varphi(t)$ is in Ξ for all terms t.

2 Traditional Proof of Completeness for Uncountable Languages

We'll define sets Λ_α, Ξ_α.

(1) There is no finite $\Gamma_0 \subseteq \Lambda_\alpha$ and finite $\Delta_0 \subseteq \Xi_\alpha$ such that $\Gamma_0 \to \Delta_0$ has a proof.

(2) $\Gamma \subseteq \Lambda_\alpha$

Set $\kappa = \{0, 1, 2, ..., \alpha, ...\}$

Where A is an L-sentence and t an L-term, enumerate all pairs $\langle A, t\rangle$:

$\langle A_0, t_0\rangle$, $\langle A_0, t_1\rangle$, ... ,$\langle A_1, t_0\rangle$, ... , $\langle A_\alpha, t_\alpha\rangle$, ...

such that for all pairs $\langle A, t\rangle$ and all $\beta \in \kappa$ there exists $\alpha > \beta$ such that $\langle A, t\rangle = \langle A_\alpha, t_\alpha\rangle$.

For this it is sufficient to assume that each $\langle A, t\rangle$ appears κ-many times.

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Define Λ, Ξ satisfying (1) and (2)

Initially $\Lambda_{-1} = \Gamma$ and Ξ_{-1} is $\emptyset$

Suppose Λ_β, Ξ_β are defined for all $\beta < \alpha$

Let $\Lambda^-_\alpha = \bigcup_{\beta<\alpha} \Lambda_\beta$ $(= \Lambda_{\alpha-1}$ if $_{\alpha-1}$ exists)

$\Xi^-_\alpha = \bigcup_{\beta<\alpha} \Xi_\beta$

Skip step (1). Do step (2) and (3)

Step (2)

If $A_\alpha \in \Lambda^-_\alpha$, and A_α is $\varphi \wedge \psi$,
put $\Lambda_\alpha = \Lambda^-_\alpha \ \cup\ \{\varphi, \psi\}$ and $\Xi_\alpha = \Xi^-_\alpha$

If $\exists\ \ \Gamma_0 \subseteq \Lambda_\alpha,\ \Delta_0 \subseteq \Xi_\alpha$ such that $\Gamma_0 \to \Delta_0$ has a proof,
then $\Gamma_0 = \Gamma^-_o \cup \{\varphi, \psi\}$

$$\frac{\Gamma^-_0, \varphi, \psi \to \Delta_0}{\Gamma^-_0, \varphi \wedge \psi \to \Delta_0}$$
Where $\Gamma^-_0, \varphi \wedge \psi \subseteq \Lambda_\beta$ for some $_{\alpha<\beta}$ and $\Delta_0 \subseteq \Xi_\beta$ for some $_\beta$

Step (3)

If $A_\alpha \in \Lambda^-_\alpha$, and A_α is $\varphi \vee \psi$,

put either

$\Lambda_\alpha = \Lambda^-_\alpha \ \cup\ \{\varphi\}$ and $\Xi_\alpha = \Xi^-_\alpha$

or

$\Lambda_\alpha = \Lambda_\alpha^- \ \cup \ \{\psi\}$ and $\Xi_\alpha = \Xi_\alpha^-$

...Whichever satisfies condition (1)

Finally, set $\Lambda = \bigcup_{\beta<\kappa} \Lambda_\beta$ and $\Xi = \bigcup_{\beta<\kappa} \Xi_\beta$

So $\exists$ a model M such that

let $\sigma(c) = c$
or $\sigma(c) = [c]$

Then $M \models \lambda[c]$ for all $\lambda \in \Lambda$
and $M \not\models \{[c]$ for all $\} \in \Xi$

So M shows Γ is satisfiable.