Math 260A — Mathematical Logic — Scribe Notes
UCSD — Winter Quarter 2012
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An Application of the Łoś-Vaught Theorem (cont.)

Recall from last time the theory $\mathrm{Th}(\mathbb{N}, 0, S)$ and the theory T, defined as follows:

$$\begin{aligned} T = \{&\forall x \forall y (S(y) = S(x) \leftrightarrow x = y), \\ &\forall x (x \neq 0 \leftrightarrow \exists y (S(y) = x)), \\ &\forall x (S(x) \neq x), \\ &\vdots \\ &\forall x (\underbrace{S(S(\cdots S}_{k}(x) \cdots)) \neq x), \\ &\vdots \\ \}& \end{aligned}$$

where the last sentence appears once for each $k \geq 1$. As we saw last time, there are two models of T with cardinality $\aleph_0$: the standard model $\mathcal{N}_1 = (\mathbb{N}, 0, S)$ and the $\mathbb{Z}$-chain model, $\mathcal{N}_2$.

Claim 1. $\mathcal{N}_1 \equiv \mathcal{N}_2$.

Theorem 1. *T is κ-categorical for any $\kappa > \aleph_0$.*

Proof. Observe that any model $\mathcal{M}$ such that $\mathcal{M} \models T$ has to contain a copy of $\mathbb{N}$, some disjoint $\mathbb{Z}$-chains, and nothing else. The isomorphism class of $\mathcal{M}$ is then determined by the number of $\mathbb{Z}$-chains it contains. In other words, apply $S^{\mathcal{M}}$ to $0^{\mathcal{M}}$ to generate a copy of $\mathbb{N}$ with no loops:

$$0^{\mathcal{M}} \xrightarrow{S^{\mathcal{M}}} 1 \xrightarrow{S^{\mathcal{M}}} 2 \xrightarrow{S^{\mathcal{M}}} \cdots$$

This accounts for $\aleph_0$ many of the elements of $|\mathcal{M}|$.

Take one of the remaining elements, call it α, and do the same thing. However, since $\alpha \neq 0$, by the second axiom there is some element $\alpha - 1$ such that $S(\alpha - 1) = \alpha$. So the chain extends in both directions, making it a $\mathbb{Z}$-chain:

$$\cdots \xrightarrow{S^{\mathcal{M}}} \alpha - 1 \xrightarrow{S^{\mathcal{M}}} \alpha \xrightarrow{S^{\mathcal{M}}} \alpha + 1 \xrightarrow{S^{\mathcal{M}}} \cdots$$

Again, take any other element α' and do the same. The $\mathbb{Z}$-chain generated by α' can't intersect the $\mathbb{Z}$-chain generated by α at any point, since every element has a unique predecessor and a unique successor, and α' doesn't lie on α's $\mathbb{Z}$-chain. You can do this κ many times, so the cardinality of $\mathcal{M}$ is $\aleph_0 + \kappa \cdot \aleph_0 = \kappa$. Note that when $\kappa = \aleph_0$, the cardinality of $\mathcal{M}$ is $\aleph_0$, and $\mathcal{M}$ is also a model of $\mathrm{Th}(\mathbb{N}, 0, S)$. Since this $\mathcal{M} \ncong (\mathbb{N}, 0, S)$, T is not $\aleph_0$-categorical, as we've seen.

However, if $\mathcal{M}$ and $\mathcal{M}'$ are two models of T that have the same cardinality ($> \aleph_0$), they're isomorphic, since any 2 $\mathbb{Z}$-chains are isomorphic. Hence T is κ-categorical. □

Corollary 1. *T is complete.*

Proof. T has no finite models, since any model of T contains a copy of $\mathbb{N}$. T is κ-categorical for $\kappa > \aleph_0$, and the language has cardinality $\aleph_0$. So by the Łoś-Vaught Test, T is complete. □

Corollary 2. *$T \models Th(\mathbb{N}, 0, S)$.*

Proof. Clearly $T \subseteq \mathrm{Th}(\mathbb{N}, 0, S)$, since all of the statements in T are true of $(\mathbb{N}, 0, S)$. As we've seen, $\mathrm{Th}(\mathbb{N}, 0, S)$ is consistent and complete. But for all sentences φ, either $T \models \varphi$ or $T \models \neg\varphi$, and T can only imply things that are consistent with $\mathrm{Th}(\mathbb{N}, 0, S)$. Since T is complete, it implies everything that is consistent with $\mathrm{Th}(\mathbb{N}, 0, S)$, which is $\mathrm{Th}(\mathbb{N}, 0, S)$ itself. □

Corollary 3. *$Cn(T) = Th(\mathbb{N}, 0, S)$*

Corollary 4. *$\mathcal{N}_1 \equiv \mathcal{N}_2$. That is,*

$$\mathcal{N}_1 \models \varphi \Leftrightarrow \varphi \in Th(\mathbb{N}, 0, S)$$
$$\mathcal{N}_2 \models \varphi \Leftrightarrow \varphi \in Th(\mathbb{N}, 0, S)$$

Nonstandard Integers

It was said that, in the model of $(\mathbb{N}, 0, S)$ with extra elements, the extra elements (the nonstandard integers) were in some sense "larger" than all of the standard integers. Since the nonstandard integers are inaccessible from the standard integers by S, what could it mean to say that they are larger?

Consider such a model in a language that includes $+$ and $\cdot$.

As we saw, from axiom 2 of T α has a chain of predecessors, but it does not include 0. We can define $x < y$ in the following way:

$$x < y := \exists z(x + S(z) = y)$$

Then $\alpha > 0$, since $0+S(\alpha-1) = \alpha$, and $(\alpha-1) < \alpha$, since $(\alpha-1)+S(0) = \alpha$.

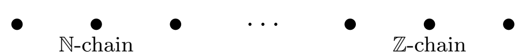
Since the model satisfies $\mathrm{Th}(\mathbb{N})$, we can say other things about α, like whether it is even or odd:

$$\exists z((z + z = \alpha) \vee (z + z + 1 = \alpha)).$$

Let $z = \frac{1}{2}\alpha$ and $z' = \frac{1}{3}\alpha$, and suppose that α is a multiple of 6. Then $3z' = 2z = \alpha$. If z' and z belong to the same $\mathbb{Z}$-chain, then $z' - z$ is some number. Suppose it is 17; then $z' = z - 17$, and we can use this and $3z' = 2z = \alpha$ to find α, which is some finite number. But since we know α isn't finite, $\frac{1}{2}\alpha$, $\frac{1}{3}\alpha$, and α all must live in different chains. So must $\sqrt{\alpha}$, which is smaller than $\frac{1}{n}\alpha$ for all n in the $\mathbb{N}$-chain.

Theorem 2. *There is no way to give a procedure to specify a full model of* $(\mathbb{N}, 0, S, +, \cdot)$. *There must be one, by completeness, but we can't specify an algorithmic procedure to construct it. That is, it has no* computable *model.*

Returning to $\mathrm{Th}(\mathbb{N}, 0, S)$, we *do* get a computable model:



which we can specify as $\{(0, i) \mid i \in \mathbb{N}\} \cup \{(1, i) \mid i \in \mathbb{Z}\}$. then

$$S((0, i)) = (0, i + 1)$$
$$S((1, i)) = (1, i + 1)$$

This model satisfies $\mathrm{Th}(\mathbb{N}, 0, S)$, which isn't hard to prove, but we don't have the tools yet. Stay tuned next quarter for the exciting conclusion!

Skolem's Paradox

Consider set theory. More specifically, consider the language $\{\in, =\}$ of set membership and equality. The following comprise a subset of the axioms of set theory:

Extensionality $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$.

Union (weak form) $\forall x \forall y \exists z (\forall w (w \in z \leftrightarrow w \in x \vee w \in y))$.

The stronger form of this axiom is about the existence of a union of a set of sets.

We can define integers using Von Neumann's definition:

$$\begin{aligned}
0 &= \emptyset \\
1 &= \{\emptyset\} = \{0\} \\
2 &= \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 &= \{\emptyset, \{\emptyset, \{\emptyset\}\}\} = \{0, 1\} \\
&\vdots \\
n &= \{0, 1, \ldots, n-1\} \\
&\vdots \\
\omega &= \{0, 1, 2, \ldots\}
\end{aligned}$$

Where ω is the set of all integers. You can write out axioms specifying what it means to be an integer and what it means to be the smallest infinite set (ω).

Ordered pairs are given by sets: $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$. Functions, then, are sets of ordered pairs. The power set of a set x is the set of functions from x to the set 2.[1]

Theorem 3. *There exists an uncountable set.*

Consider the particular axiomatization given by Zermelo-Fraenkel set theory with Choice (ZFC) which is presumably consistent, hence has a model.[2] By the downward Löwenheim-Skolem theorem, then, it has a countable model. But this is weird, since by Theorem 3 it contains an uncountable set. We also have

Theorem 4. *There is no largest cardinal.*

So the set that we usually call $\mathbb{R}$ is described by ZFC, and must be contained in its countable model $\mathcal{M}$. Quasi-formally,

$$\mathcal{M} \models \exists \mathbb{R} [\neg \exists \text{ a 1–1 function } f \colon \mathbb{R} \to \mathbb{N}].$$

[1] Prof. Buss didn't give a definition of power set, though he wrote "power set" on the board. This seemed like the obviously intended definition, given the context. — JD

[2] It can only *prove* itself consistent if it is inconsistent, which see Gödel's second incompleteness theorem.

In "real life" we know that this sentence is true, but it's not clear how it could be true in $\mathcal{M}$. Note that, trivially,

$$\mathbb{R}^{\mathcal{M}} = \{m \in |\mathcal{M}| \mid m \in^{\mathcal{M}} \mathbb{R}^{\mathcal{M}}\} \subseteq |\mathcal{M}|,$$

which is countable. Given all of the functions in mathematics, there must be one sending this $\mathbb{R}$ to $\mathbb{N}$ bijectively. The resolution to the paradox is noting that this function f doesn't exist in $\mathcal{M}$, though it does in "real life." The paradox is generated by not observing the distinction between the object language of the model of ZFC and the metalanguage of mathematical practice.

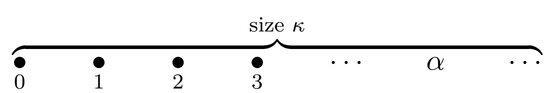
Other zany consequences obtain. There's even another model of ZFC in $\mathcal{M}$. We could also add $c \in |\mathcal{M}|$ and the axioms $\{c \neq 0, c \neq 1, \dots\}$ and get nonstandard integers.

Completeness for uncountable languages ($\kappa > \aleph_0$)

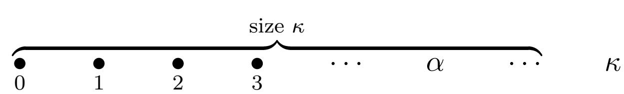
Given the Axiom of Choice, we have the following fact about cardinals:

Theorem 5. *Any set can be well-ordered.*

In fact, we view a cardinal κ as a well-ordering of cardinality κ. Take a canonical set K of size κ:



This can be done so that no proper initial ordering has cardinality κ. Stick a new element on top:



The set is still a well-ordering, and it still has cardinality κ. So there must be some first element such that all larger well-orders have cardinality κ and all proper initial segments are smaller than κ. Then κ is the well-order such that no proper initial segment has size κ.