Math 260A — Mathematical Logic — Scribe Notes
UCSD — Winter Quarter 2012
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Monday, February 27, 2012

1 Löwenheim-Skolem Theorems

Recall the following from last lecture:

Theorem 1. *If a set of sentences T is consistent, in a language L of cardinality κ, then T has a model of cardinality $\leq \{\aleph_0, \kappa\}$*

Corollary 1 (The Downward Löwenheim-Skolem Theorem)**.** *Suppose a language L has cardinality κ, and a set of sentences T has a model of cardinality $\kappa' > \kappa$, then T has a model of cardinality $max\{\aleph_0, \kappa\}$, assuming κ' infinite.*

Proof. Enlarge language L to

$$L' = L \cup \{C_\alpha : \alpha < max\{\aleph_0, \kappa\}\}$$

Where C_α's are $max\{\aleph_0, \kappa\}$ many new constant symbols. Set

$$T' := T \cup \{C_\alpha \neq C_\beta : \alpha \neq \beta\}$$

T' has a model of cardinality κ', and T' has a model of cardinality $\leq max\{\aleph_0, \kappa\}$ ($\kappa < \aleph_0$ L'-cardinality $\aleph_0$)

T' has no models of cardinality $< max\{\aleph_0, \kappa\}$ since the C_α's have to have distinct interpretations. Therefore, T must have a model of cardinality $max\{\aleph_0, \kappa\}$. □

Theorem 2 (The Upperward Löwenheim-Skolem Theorem)**.** *If T has an infinite model, then for all $\kappa' \geq max\{\aleph_0, \kappa\}$, T has a model of cardinality κ'.*

Proof. Similar to the Downward Lowenheim-Skolem Theorem, but note every finite subset of

$$T \cup \{C_\alpha \neq C_\beta : \alpha \neq \beta < max\{\aleph_0, \kappa\}\}$$

is consistent. □

Can replace the hypothesis that T has an infinite model with "T has models of arbitrarily large finite size". That is, $\forall n \; \exists m \geq n$, T has a model of size m $(m, n \in \mathbb{N})$.

Definition 1. T is **categorical** iff any two models of T are isomorphic.

Definition 2. T is **κ-categorical** iff T has only one model of cardinality κ, up to isomorphism.

Consider a finite structure $\mathcal{M}$, $|\mathcal{M}| = n$, and suppose language of $\mathcal{M}$ is finite.

Theorem 3. *There exists a single sentence φ such that the only model of φ is $\mathcal{M}$, up to isomorphism.*

Proof. φ is:

$$(\exists x_1 \exists x_2 \dots \exists x_n)[\forall y(y = x_1 \vee y = x_2 \vee \cdots \vee y = x_n)$$
$$\wedge (\bigwedge_{i<j} x_i \neq x_j)$$
$$\wedge (\bigwedge \text{giving all values of } f)$$
$$\wedge (\bigwedge \text{giving all values of } P)]$$

□

Note that φ is categorical.

2 Łoś -Vaught

If a set of sentences T is κ'-categorical and has no finite models, where $\kappa' \geq \kappa$ (κ-cardinality of the language), then T is complete.

Recall for Th a theory, Th is complete iff for all sentences φ, $\varphi \in Th$ or $\neg\varphi \in Th$.

Definition 3. For T a set of sentences, T is complete iff for all sentences φ, $T \models \varphi$ or $T \models \neg\varphi$ (Think of T as a set of axioms, identifying this with $Cn(T)$).

Proof. Suppose T is not complete, so $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are both consistent, for some sentence φ.

Lemma 1. *Let T be a set of sentences, φ a sentence, $T \models \varphi$ iff $T \cup \{\neg\varphi\}$ is not consistent.*

Take model $\mathcal{M}_1 \models T \cup \{\varphi\}$ and model $\mathcal{M}_2 \models T \cup \{\neg\varphi\}$. By Upperward Lowenheim-Skolem Theorem, $\mathcal{M}_1$ and $\mathcal{M}_2$ can be chosen to have cardinality κ'.

So $\mathcal{M}_1 \cong \mathcal{M}_2$ and we have a contradiction.

$\square$

Example 1.

$$T = \{\exists x \forall y (x = x) \vee G_k : k \leq 1\}$$

T is $\aleph_0$-categorical. $T = \{a\}$, $T \models \exists x \forall y (x = y)$, $T \not\models \neg\exists x \forall y (x = y)$

$$G_k : \exists \geq k \text{ element}$$

$$G_k := \exists x_1 \dots \exists x_k (\bigwedge_{i<j} x_i \neq x_j)$$

Let T be $\{G_k : k \geq 2\}$. T is κ-categorical for all κ, so T is complete.

Example 2. Let language $L = \{0, S\}$, the idea is $(\mathbb{N}, 0, S)$.

Axioms:

1. $\forall x (\exists y (S(y) = x) \leftrightarrow x \neq 0)$
2. $\forall x \forall y (S(x) = S(y) \leftarrow x = y)$
3. $\forall x (S(S(\dots S(x) \dots)) \neq x)$ for all $k \geq 1$

Let $T =$ Set of these axioms.

Claim 1. *T is κ-categorical for all $\kappa > \aleph_0$ but T is not $\aleph_0$-categorical.*

If this claim is correct, then T is complete, and also $T \models Th(\mathbb{N}, 0, S)$.