Math 260A Homework 3

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In the previous lecture we proved there is no set Γ of sentences in any language which characterizes equality. That is to say such that if $\mathcal{M} \vDash \Gamma$ then $\mathcal{M}$ is finite. Conversely recall from homework 2 problem 12 there does exist a set Γ of sentences such that $\mathcal{M} \vDash \Gamma$ if and only if $\mathcal{M}$ is infinite. To see this let $G_k : \exists x_1 \exists \ldots \exists x_k (\bigwedge_{i<j} x_i \neq x_j)$ and $L_k : \exists x_1 \exists \ldots \exists x_k \forall y (\bigvee_{i<j} y = x_k)$ observe $\neg L_k \iff g_{k+1}$. Letting $\Gamma = \{G_k\}_{k \geq 0}$ satisfies our requirement. Note that there is no way to express a compliment of Γ

Definition 1. *For a structure $\mathcal{M}$ the* ***size*** *or* ***cardinality*** *of the structure refers to the size of $|\mathcal{M}|$*

Definition 2. *The* ***theory*** *of a structure $\mathcal{N}$, $Th(\mathcal{N})$ denotes the set of sentences true in $\mathcal{N}$. That is $Th(\mathcal{N}) = \{\varphi : \mathcal{N} \vDash, \varphi$ a sentence$)$*

Definition 3. *A theory T is a set of sentences closed under logical consequence. That is to say if $T \vDash \varphi$ where φ is a sentence then $\varphi \in T$.*

Definition 4. ***Consequence*** *is denoted $Cn(T) = \{\varphi : T \vDash \Phi, \varphi$ a sentence$\}$*

Definition 5. *A theory T is* ***complete*** *if for all sentences $\varphi \in T$ or $\neg\varphi \in T$.*

Theorem 1. *There exists a model $\mathcal{M}$ such that $\mathcal{M} \ncong \mathcal{N}$ and $\mathcal{M} \vDash Th(\mathcal{N})$*

Definition 6. *If $\mathcal{M}$ is* ***isomorphic*** *to $\mathcal{N}$ denoted $\mathcal{M} \cong \mathcal{N}$ then there exists a bijection $h : |\mathcal{M}| \to |\mathcal{N}|$ such that $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$. For all $m_1, m_2, \ldots m_k \in |\mathcal{M}|$, $< m_1, m_2, \ldots m_k > \in P^{\mathcal{M}}$ iff $< h(m_1), h(m_2), \ldots h(m_k) > \in P^{\mathcal{N}}$. Also $h(f^{\mathcal{M}}(m_1, \ldots m_k)) = f^{\mathcal{N}}(h(m_1), \ldots h(m_k))$.*

Definition 7. *$\mathcal{M}$ is* ***elementarily equivalent*** *to $\mathcal{N}$ denoted $\mathcal{M} \equiv \mathcal{N}$ if and only if for all sentences φ, $\mathcal{M} \vDash$ iff $\mathcal{N} \vDash \varphi$*

Before proving the theorem stated above, we shall prove a specific case of it.

Theorem 2. *There exists a structure with the same propositions as $\mathbb{N}$ but not $\mathbb{N}$.*

Proof. Let L' be the language with model $(\mathbb{N}, 0, S, +, \cdot, c)$. The same as the model $\mathcal{N}$ (with language L) but with a new constant c added. Let $T' := Th(\mathcal{N}) \cup \{\neg(c = 0), \neg(c = S(0)), \neg(c = S(S(0))), \ldots\}$ I claim T' is consistent. Suppose it were not then $T'_k := Th(\mathcal{N}) \cup \{(c \neq 0), (c \neq S(0)), (c \neq S(S(0))), \ldots, c \neq S^k(0)\}$ is not consistent for some k. Where $S^k(0)$ is S applied k times. But $\mathcal{N}'_k = (\mathbb{N}, 0, S, +, \cdot, k+1)$ where $k + 1 = c^{\mathcal{N}}$ is a model for T'_k.

Since T' is consistent it has a model $\mathcal{M}'$ (by completeness). Take the restriction of $\mathcal{M}'$ to language L. Call this $\mathcal{M}$. (in original language) We will now establish $\mathcal{N} \cong \mathcal{M}$. Consider:

$$
\begin{aligned}
&h : |\mathcal{N}| \to |\mathcal{M}| \\
&h(0) = 0^{\mathcal{M}} \\
&h(1) = S^{\mathcal{M}}(0^{\mathcal{M}}) \\
&h(k) = S^{\mathcal{M}}(S^{\mathcal{M}}(\dots 0^{\mathcal{M}}))
\end{aligned}
$$

None the less $c^{\mathcal{M}'} \notin$ range of h. □

Observe that $c^{\mathcal{M}'}$ is larger than all of the integers because for any k, $\forall x(x \neq 0 \wedge x \neq 1 \dots x \neq k \to k < x)$.

Definition 8. *A theory T is **categorical** iff it has up to isomorphism a unique model. In practice this is rare*

For example $Th(\mathcal{N})$ is not categorical.

Definition 9. *A model $\mathcal{M}$ of T is a structure satisfying T.*

Definition 10. *A **cardinal** is a size of a set. The countable cardinals consist of $0, 1, 2, \dots$ as well a $|\mathbb{N}|$, ω, or $\aleph_0$. The uncountable cardinals (assuming the axiom of choice) are $\aleph_1, \aleph_2, \dots, c$*

Theorem 3. *Let κ be a cardinal. For any infinite κ there is a structure $\mathcal{M}$ of cardinality κ s.t $\mathcal{M} \equiv (\mathbb{N}, 0, S, +, \cdot)$*

Proof. Let $L' = \{0, S, +, \cdot\} \cup \{c_\alpha : \alpha \in [\kappa]\}$. $T' = Th(\mathbb{N}, 0, S, +, \cdot) \cup \{c_\alpha \neq c_\beta : \alpha \neq \beta \in [\kappa]\}$ By the compactness theorem T' is consistent and by the completeness theorem T' has a model. We can apply the same argument as before. Note the the strong form of the completeness theorem remains to be proved. □

Back to Completeness

Recall if $T \vDash \Gamma \to \Delta$ then there exists a LK-proof of $T_0, \Gamma \to \delta$, where T_0 is a finite subset of T. In our proof of completeness we assumed the language countable (enumerated all formulas) and obtained a model $\mathcal{M}$ and a σ such that $\mathcal{M} \vDash T, \Gamma[\sigma]$ and $\mathcal{M} \nvDash \Delta[\sigma]$. $\mathcal{M}$ was countable with only countably many terms.

Theorem 4. *If T is consistent, or equivalently no LK proof of $T_0 \to$ exists when T_0 is a finite subset of T, then T is a countable model provided the language is countable.*

Theorem 5. *If T is consistent in a language of cardinality κ then T has a model of cardinality less then or equal to κ.*