

Math 260A — Mathematical Logic — Scribe Notes
UCSD — Winter Quarter 2012
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1 Language with equality

Recall that in the previous class, we proved the Completeness Theorem (without equality), that is that if $T \models \Gamma \rightarrow \Delta$, then $\Pi, \Gamma \rightarrow \Delta$. Also, remember the following equality axioms for any terms s, t, u, s_i, t_i :

- $\rightarrow s = s$
- $s = t \rightarrow t = s$
- $s = t, t = u \rightarrow s = u$
- $s_1 = t_1, \dots, s_k = t_k \rightarrow f(s_1, \dots, s_k) = f(t_1, \dots, t_k)$ for any formula f of arity k .
- $s_1 = t_1, \dots, s_k = t_k \rightarrow P(s_1, \dots, s_k) \rightarrow P(t_1, \dots, t_k)$ for any relation P of arity k .

1.1 The Completeness Theorem with Equality

There are a couple of problems which would occur if we added the equality axioms to our earlier proof of the Completeness Theorem without modification. One is that if \mathcal{S} is the set of equality axioms, we would have to add their closure to the conclusion ($\forall \mathcal{S}, \Pi, \Gamma \rightarrow \Delta$.) To discuss the other problem, first recall that in the proof of the Completeness Theorem, we considered:

- $\Delta \rightarrow \exists$;
- $|\mathcal{M}|$ the set of terms; and
- $c^{\mathcal{M}}, f^{\mathcal{M}}$, and $P^{\mathcal{M}}$.

Then if we consider equality, we might have $s = t$ in Δ for distinct terms s and t . This would define some equality $s =^{\mathcal{M}} t$ which would no longer be true equality. In effect, equality would become a predicate symbol. Thus we should modify the proof of the Completeness Theorem when we add equality.

We now describe how to modify the previous proof of the Completeness Theorem #1 and #3 a, to prove Completeness Theorems #2 and #3 b, (using our previous numbering system). First, must modify our definition of active.

Definition 1. A sequent $\Pi, \Gamma' \rightarrow \Delta'$ is active provided that:

- $(\Pi \cup \Gamma') \cap \Delta = \emptyset$
- There is no equality axiom $E \rightarrow E'$ such that $E \subset \Pi, \Gamma'$ and $E' \subset \Delta'$.

As an example of the latter, for a sequent to be active, if

$$s_1 = t_1, \dots, s_k = t_k, P(\bar{s}) \in \Gamma',$$

then $P(\bar{t}) \notin \Delta'$.

Additionally, there is a new case (g), in particular: If A is $s = t$, for each active sequent $\Pi, \Gamma' \rightarrow \Delta'$, then

$$\frac{\Pi, \Gamma' \rightarrow \Delta', s = t \quad s = t, \Pi, \Gamma' \rightarrow \Delta'}{\Pi, \Gamma' \rightarrow \Delta'}$$

Next, we must modify the end of our proof. As before, if the process stops in finitely many steps, we have (finite) proof, since we have cuts. Otherwise, again find an infinite branch and Δ and Ξ as before which satisfy the same properties plus the additional extra property that: For all formulas $s = t$, either $s = t \in \Delta$ or $s = t \in \Xi$.

Fix Δ and Ξ . Define a relation $s \sim t$ if and only if $s = t$ is in Δ . In fact, this is an equivalence relation.

Theorem 1. For all s , $s = s$ is in Δ .

Proof. We have the equality axiom $\rightarrow s = s$, so $s = s$ is not in Ξ or our sequent is inactive. Thus $s = s$ is in Δ . □

Theorem 2. $s \sim t$ if and only if $t \sim s$.

Proof. Similarly, we can't have one of $s = t$ and $t = s$ in each of Δ and Ξ , since we have the axioms $s = t \rightarrow t = s$ and $t = s \rightarrow s = t$. □

Theorem 3. If $s \sim t$ and $t \sim u$ then $s \sim u$.

Proof. $s = t$, $t = u$, and $s = u$ are each in Δ or Ξ . For the theorem to fail, $s = t$ and $t = u$ would both be in Δ , but $s = u$ would be in Ξ , but again this would contradict the fact that we have an active sequent. □

We will want another similar theorem:

Theorem 4. *If $s_1 \sim t_1 \dots s_k \sim t_k$, then $f(\bar{s}) = f(\bar{t})$.*

Proof. Similarly, the theorem fails if $s_i = t_i$ is in Δ for all i , but $f(\bar{s}) = f(\bar{t})$ is in Ξ and hence not in Δ , but again this would contradict the fact that we have an active sequent. \square

Definition 2. $[s] = \{t : s \sim t\}$

Define the following:

- $|\mathcal{M}| = \{[s] : s \text{ appears as a term in } \Delta, \Xi\}$.
- $c^{\mathcal{M}} = [c]$
- $f^{\mathcal{M}}([s_1], \dots, [s_k]) = [f(s_1, \dots, s_k)]$. (Note that this is well defined by the previous theorem.)
- $=$ as true equality
- $\langle [s_1], \dots, [s_k] \rangle \in P^{\mathcal{M}}$ if and only if there exists t_1, \dots, t_k such that $t_i \sim s_i$ for all i and $P(t_1, \dots, t_k) \in \Delta$.

(Note that the last requirement is necessarily complicated. If we have $s = t$ and $P(t)$ in Δ , then we want to have $P^{\mathcal{M}}([t])$ or $P^{\mathcal{M}}([s])$.) Note the following trivial theorem:

Theorem 5. *If $s_i \sim t_i$ for all i and $P^{\mathcal{M}}([s_1], \dots, [s_k])$, then $P^{\mathcal{M}}([t_1], \dots, [t_k])$*

Also,

Theorem 6. *If ϕ is an atomic formula in Ξ and ϕ has the form $P(s_1, \dots, s_k)$ then $\mathcal{M} \not\models P^{\mathcal{M}}([s_1], \dots, [s_k])$. If ϕ has the form $s = t$ then $[s] \neq [t]$.*

Proof. For $s = t$, $s = t \notin \Delta$ so $[s] \neq [t]$. If $P^{\mathcal{M}}([s_1], \dots, [s_k])$ then there exists t_i , $t_i \sim s_i$ such that $P(t_1, \dots, t_k) \in \Delta$. Then $t_i = s_i$ is in Δ for all i , $P(t_1, \dots, t_k) \in \Delta$, but $P(s_1, \dots, s_k) \in \Xi$, a contradiction. \square

Claim 1. *Let $\sigma : a \rightarrow [a]$ for any free variable a , then for all $\Phi \in \Delta$, $\mathcal{M} \models \phi[\sigma]$. For all $\phi \in \Xi$, $\mathcal{M} \not\models \phi[\sigma]$.*

Proof. The proof is as before with a modified Ξ . Note that we have it for atomic formulas as before and the other cases are the same proof. It requires the lemma that $t[\sigma] = [t]$. \square

Recall the definition for $t[\sigma]$, where $a[\sigma] = \sigma(a)$, $c[\sigma] = c^{\mathcal{M}}$, and

$$f(s_1, \dots, s_k)[\sigma] = f^{\mathcal{M}}(s_1[\sigma], \dots, s_k[\sigma]).$$

Note: Before if we were working with quantifiers, we always had infinite models by our construction. Here our model may be finite because of our equivalence relation.

1.2 The Compactness Theorem

As a corollary we have the compactness theorem.

Corollary 1 (The Compactness Theorem v.1). *Let Γ be a set of sentences. Then Γ is satisfiable if and only if each finite subset of Γ is satisfiable.*

Proof. Recall that Γ is satisfiable if and only if there exists a structure \mathcal{M} such that $\mathcal{M} \models \Gamma$. The proof in the forward direction is obvious, so we prove the remainder. Suppose Γ is not satisfiable. Then

$$\Gamma \not\models \text{.}$$

(If you prefer, we could use $\Gamma \models \perp$.) So there exists a finite $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \not\models \text{.}$ By the Soundness Theorem, there is no model \mathcal{M} satisfying Γ_0 so Γ_0 is unsatisfiable. \square

Corollary 2 (The Compactness Theorem v.2). *Let Γ be a set of sentences. Then $\Gamma \models \phi$ if and only if there exists a finite $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \models \phi$.*

Definition 3. A structure \mathcal{M} is finite if $|\mathcal{M}|$ is finite.

An application of the previous theorems is the following:

Corollary 3. *There is no set Γ of sentences such that Γ is true in precisely the finite structures.*

Proof. By contradiction. Let L be a language of Γ and

$$L' = L \cup \{c_1, \dots, c_i, \dots, i \in \mathbb{N}\},$$

where the c_i are new constant symbols. Form $\Gamma' = \Gamma \cup \{c_i \neq c_j\}_{i < j}$.

Then Γ' is consistent (satisfiable) since each finite $\Gamma'_0 \subset \Gamma'$ is satisfiable. So Γ' has a model $\mathcal{M} \models \Gamma'$. Then restrict \mathcal{M} to the language L and we must have that $\mathcal{M} \models \Gamma$. \square