Math 260A — Mathematical Logic — Scribe Notes
UCSD — Winter Quarter 2012
Instructor: Sam Buss

Notes by: Tomoya Sato
Friday, February 17, 2012

The Completeness Theorem for LK

Last time, a procedure to construct an LK-proof of $\Pi, \Gamma \longrightarrow \Delta$ was given. This time, we show that the procedure will halt after finitely many steps if $T \vDash \Gamma \longrightarrow \Delta$.

WTS

Suppose that the backward process goes on forever and we get an infinite tree. Our goal is to build a structure $\mathcal{M}$ such that $\mathcal{M} \vDash T$ and $\Gamma \longrightarrow \Delta$ is not valid in $\mathcal{M}$, i.e., there is an object assignment σ such that

$$\mathcal{M} \vDash \gamma[\sigma] \quad \forall \gamma \in \Gamma,$$
$$\mathcal{M} \nvDash \delta[\sigma] \quad \forall \delta \in \Delta.$$

Structure $\mathcal{M}$

Choose an infinite branch in the infinite tree. Let Λ be a set of formulas that appear in some *antecedent* along the infinite path, and Ξ a set of formulas that appear in some *succedent* along the infinite path.

$\mathcal{M}$ is defined as follows:

- $|\mathcal{M}| := \{\tilde{t} : t$ is a term (that includes only free variables) that appears in $\Lambda, \Xi\}$;
- $\langle \tilde{t_1}, \ldots, \tilde{t_k} \rangle \in P^{\mathcal{M}} \overset{\text{def}}{\iff} P(t_1, \ldots, t_k) \in \Lambda \overset{\text{alt}}{\iff} P(t_1, \ldots, t_k) \notin \Xi$;
- $c^{\mathcal{M}} := \tilde{c}$
- $f^{\mathcal{M}}(\tilde{t_1}, \ldots, \tilde{t_k}) = f(\widetilde{t_1, \ldots,} t_k)$. (`\widetilde`... not so beautiful...)

Our proof idea is that Λ defines truth in $\mathcal{M}$, while Ξ defines falsity in $\mathcal{M}$.

Properties of Λ

1. If $\neg A \in \Lambda$, then $A \in \Xi$;
2. If $A \wedge B \in \Lambda$, then $A \in \Lambda$ and $B \in \Lambda$;
3. If $A \vee B \in \Lambda$, then $A \in \Lambda$ or $B \in \Lambda$;
4. If $A \to B \in \Lambda$, then $A \in \Xi$ or $B \in \Lambda$;
5. If $\forall x A(x) \in \Lambda$, then for all terms t, $A(t) \in \Lambda$;
6. If $\exists x A(x) \in \Lambda$, then for some free variable b, $A(b) \in \Lambda$;

For example,

$$\frac{B{\to}C, \Gamma' \longrightarrow \Delta', B \qquad C, B{\to}C, \Gamma' \longrightarrow \Delta'}{B{\to}C, \Gamma' \longrightarrow \Delta'}$$

exemplifies 4 above, and

$$\frac{B(c), \exists x B(x), \Gamma' \longrightarrow \Delta'}{\exists x B(x), \Gamma' \longrightarrow \Delta'}$$

exemplifies 6.

Properties of Ξ

1. If $\neg A \in \Xi$, then $A \in \Lambda$;
2. If $A \wedge B \in \Xi$, then $A \in \Xi$ or $B \in \Xi$;
3. If $A \vee B \in \Xi$, then $A \in \Xi$ and $B \in \Xi$;
4. If $A \to B \in \Xi$, then $A \in \Lambda$ and $B \in \Xi$;
5. If $\forall x A(x) \in \Xi$, then for some b, $A(b) \in \Xi$;
6. If $\exists x A(x) \in \Xi$, then for all t, $A(t) \in \Xi$.

Main Claim

Lemma Let id be an object assignment such that for all c, $\mathsf{id}(c) := \tilde{c}$. Then, for all formulas A,

(a) If $A \in \Lambda$, $\mathcal{M} \vDash A[\mathsf{id}]$;

(b) If $A \in \Xi$, $\mathcal{M} \nvDash A[\mathsf{id}]$.

Since $T, \Gamma \subseteq \Lambda$ and $\Delta \subseteq \Xi$, this suffices to prove the completeness theorem. We prove the Lemma by induction of the complexity of A. But before that...

Sublemma For all t, $\mathsf{id}(t) = \tilde{t}$.

(Base Case): If t is a free variable b, $\mathsf{id}(b) = \tilde{b}$. If t is a constant symbol c, $\mathsf{id}(c) = c^{\mathcal{M}} = \tilde{c}$.
(Induction Step): If t is $f(t_1, \ldots, t_k)$,
$\mathsf{id}(f(t_1, \ldots, t_k)) = f^{\mathcal{M}}(\mathsf{id}(t_1), \ldots, \mathsf{id}(t_k)) = f^{\mathcal{M}}(\tilde{t_1}, \ldots, \tilde{t_k}) = f(\widetilde{t_1, \ldots}, t_k)$.

Proof of the Lemma

(Base Case): If A is atomic, A is $P(t_1, \ldots, t_k)$.

$$\begin{aligned} \mathcal{M} \vDash A[\mathsf{id}] &\overset{\text{iff}}{\Longleftrightarrow} \mathcal{M} \vDash P(t_1, \ldots, t_k)[\mathsf{id}] \\ &\overset{\text{iff}}{\Longleftrightarrow} \langle \mathsf{id}(t_1), \ldots, \mathsf{id}(t_1) \rangle \in P^{\mathcal{M}} \\ &\overset{\text{iff}}{\Longleftrightarrow} \langle \tilde{t_1}, \ldots, \tilde{t_1} \rangle \in P^{\mathcal{M}} \\ &\overset{\text{iff}}{\Longleftrightarrow} P(t_1, \ldots, t_k) \in \Lambda \end{aligned}$$

If $P(t_1, \ldots, t_k) \in \Xi$, then $P(t_1, \ldots, t_k) \notin \Lambda$. So, $\mathcal{M} \nvDash A[\mathsf{id}]$.

(Induction Step):

(a) A is $\neg B$: If $A \in \Lambda$, then $B \in \Xi$. By induction hypothesis, $\mathcal{M} \nvDash B[\mathsf{id}]$. So, $\mathcal{M} \vDash A[\mathsf{id}]$. The similar goes for the case where $A \in \Xi$.

(b) A is $B \wedge C$: If $A \in \Lambda$, then $B \in \Lambda$ and $C \in \Lambda$. So, by induction hypothesis, $\mathcal{M} \vDash B[\mathsf{id}]$ and $\mathcal{M} \vDash C[\mathsf{id}]$. Thus, $\mathcal{M} \vDash A[\mathsf{id}]$.

If $A \in \Xi$, then $B \in \Xi$ or $C \in \Xi$. So, $\mathcal{M} \nvDash B[\mathsf{id}]$ or $\mathcal{M} \nvDash C[\mathsf{id}]$. Thus, $\mathcal{M} \nvDash A[\mathsf{id}]$.

(c) Similar arguments go for $\vee$ and $\rightarrow$.

(d) A is $\forall B(x)$: If $A \in \Lambda$, the for all terms t, $B(t) \in \Lambda$, i.e., $\mathcal{M} \vDash B(t)[\mathsf{id}]$. For any $m \in |\mathcal{M}|$, m is $\tilde{t}$ for some t. So, for any $m \in |\mathcal{M}|$, $\mathcal{M} \vDash B(x)[\mathsf{id}_{x \mapsto m}]$. Thus, $\mathcal{M} \vDash A(x)[\mathsf{id}]$.

If $A \in \Xi$, then for some variable b, $B(b) \in \Xi$. So, by induction hypothesis, $\mathcal{M} \nvDash B(b)[\mathsf{id}]$. Thus, $\mathcal{M} \nvDash A(x)[\mathsf{id}]$.

(e) The similar goes for the case in which A is $\exists B(x)$.

□