Math 260A — Mathematical Logic — Scribe Notes
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The Completeness Theorem for LK

We begin the proof of the Completeness Theorem for the first-order sequent calculus. Note that our proof only applies for countable languages without equality.

Completeness Theorem 1 *Let* $\Gamma \longrightarrow \Delta$ *be a sequent in a first-order language* L *which does not contain equality. If* $\mathbb{S}$ *is a set of sequents and* $\forall\mathbb{S} \vDash \Gamma \longrightarrow \Delta$*, then there exists a finite subset of* $\forall\mathbb{S}$*,* Π*, such that* $\Pi, \Gamma \longrightarrow \Delta$ *has an LK proof.*

We also provide an alternate formulation:

Completeness Theorem 2 *Let* $\Gamma \longrightarrow \Delta$ *be a sequent in a first-order language* L *which does not contain equality. If* T *is a set of sentences and* $T \vDash \Gamma \longrightarrow \Delta$*, then there exists a finite* $\Pi \subseteq T$ *such that* $\Pi, \Gamma \longrightarrow \Delta$ *has an LK proof.*

The idea of our proof will be to work backwards to find of proof of $\Pi, \Gamma \longrightarrow \Delta$ (we do not yet know what this Π will be, however).

We begin by enumerating all L-formulas as

$$A_1, A_2, A_3, \ldots$$

where every L-formula appears infinitely often. We can do this as follows.

Since we assume L to be countable, enumerate the functions, predicates, and constants in L:

$$\begin{gathered} f_1, f_2, f_3, \ldots \\ P_1, P_2, P_3, \ldots \\ c_1, c_2, c_3 \ldots \quad . \end{gathered}$$

Now, for $i = 1, 2, 3, \ldots$ list out all L-formulas with $\leq i$ symbols that have subscripts $\leq i$. This enumeration will guarantee that we list all L-formulas and that each L-formula will appear over and over again.

Likewise, enumerate all L-terms

$$t_1, t_2, t_3, \ldots \quad .$$

And then enumerate all formula-term pairs, $< A_i, t_j >$, in a list where again each pair appears infinitely often. FOr example, we can use a loop similar to the one defined above:

$$< A_1, t_1 >, < A_1, t_1 >, < A_2, t_1 >, < A_1, t_2 >, < A_2, t_2 >, \ldots \quad .$$

Now we try building a proof P. Start with $\Gamma \longrightarrow \Delta$. If $\Gamma \cap \Delta \neq \emptyset$, then it is easy to give a proof of $\Gamma \longrightarrow \Delta$ using *Weakening:left* and *Weakening:right* inferences. This motivates the following definition.

Definition The sequent $\Gamma' \longrightarrow \Delta'$ is *active* if $\Gamma' \cap \Delta' = \emptyset$.

As we build P, we will work on active sequents. Note that P will be a tree of sequents.

To begin with, P will be the single sequent $\Gamma \longrightarrow \Delta$. Take the next pair[1] $< A, t >$ in the enumeration.

Step 1: If $A \in \forall\mathbb{S}$ (or if $A \in T$ depending on which of the above formulations one prefers), add A to every antecedent in P. Π will end up being the set of such A's added by this step.[2]

Step 2: For every active sequent that contains A, update it as follows:

Case a): If A is $\neg B$ and a sequent $\neg B, \Gamma' \longrightarrow \Delta'$ is active in P, replace it by:

$$\frac{\neg B, \Gamma' \longrightarrow \Delta', B}{\neg B, \Gamma' \longrightarrow \Delta'}$$

[1] The *first* next pair is the first pair in the enumeration.

[2] Since T contains sentences, A is a sentence so we do not have to worry about contradicting eigenvariable conditions. The same holds if one prefers the $\forall\mathbb{S}$ formulation.

If A is $\neg B$ and a sequent $\Gamma' \longrightarrow \Delta', \neg B$ is active in P, replace it by:

$$\frac{B, \Gamma' \longrightarrow \Delta', \neg B}{\Gamma' \longrightarrow \Delta', \neg B}$$

Note that the upper sequent could now be inactive, but it could also still be active.

Case b): If A is $B \wedge C$, then any active sequent $B \wedge C, \Gamma' \longrightarrow \Delta'$ in P is replaced by:

$$\frac{B, C, B \wedge C, \Gamma' \longrightarrow \Delta'}{B \wedge C, \Gamma' \longrightarrow \Delta'}$$

If A is $B \wedge C$, then any active sequent $\Gamma' \longrightarrow \Delta', B \wedge C$ in P is replaced by:

$$\frac{\Gamma' \longrightarrow \Delta', B \wedge C, B \qquad \Gamma' \longrightarrow \Delta', B \wedge C, C}{\Gamma' \longrightarrow \Delta', B \wedge C}$$

Again, the upper sequents may be active or inactive.

Case c): If A is $B \vee C$, then every active sequent in P of the form $B \vee C, \Gamma' \longrightarrow \Delta'$ is replaced by:

$$\frac{B, B \vee C, \Gamma' \longrightarrow \Delta' \qquad C, B \vee C, \Gamma' \longrightarrow \Delta'}{B \vee C, \Gamma' \longrightarrow \Delta'}$$

If A is $B \vee C$, then every active sequent in P of the form $\Gamma' \longrightarrow \Delta', B \vee C$ is replaced by:

$$\frac{\Gamma' \longrightarrow \Delta', B \vee C, B, C}{\Gamma' \longrightarrow \Delta', B \vee C}$$

Case d): If A is $B \rightarrow C$, then any active sequent in P of the form $\Gamma' \longrightarrow \Delta', B \rightarrow C$ is replaced by:

$$\frac{B, \Gamma' \longrightarrow \Delta', C, B \rightarrow C}{\Gamma' \longrightarrow \Delta', B \rightarrow C}$$

If A is $B \rightarrow C$, then any active sequent in P of the form $B \rightarrow C, \Gamma' \longrightarrow \Delta'$ is replaced by:

$$\frac{B \rightarrow C, \Gamma' \longrightarrow \Delta', B \qquad C, B \rightarrow C, \Gamma' \longrightarrow \Delta'}{B \rightarrow C, \Gamma' \longrightarrow \Delta'}$$

Case e): Suppose A is $\forall x B(x)$. Let $\Gamma' \longrightarrow \Delta', \forall x B(x)$ be an active sequent, let b be some (new) free variable, not used anywhere in P yet. Then replace this with:

$$\frac{\Gamma' \longrightarrow \Delta', \forall x B(x), B(b)}{\Gamma' \longrightarrow \Delta', \forall x B(x)}$$

Note that in this case the top cedent will certainly be active since b is a completely new variable.

Likewise, if $\forall x B(x), \Gamma' \longrightarrow \Delta'$ is active then replace it by:

$$\frac{B(t), \forall x B(x), \Gamma' \longrightarrow \Delta'}{\forall x B(x), \Gamma' \longrightarrow \Delta'}$$

Note that the term t from our ordered pair $< A, t >$ finally comes into play here.

Case f): If A is of the form $\exists x B(x)$, c is a new free variable not used in P yet, and $\exists x B(x), \Gamma' \longrightarrow \Delta'$ is active in P, replace it by:

$$\frac{B(c), \exists x B(x), \Gamma' \longrightarrow \Delta'}{\exists x B(x), \Gamma' \longrightarrow \Delta'}$$

Likewise any active sequent of the form $\Gamma' \longrightarrow \Delta', \exists x B(x)$ is replaced by:

$$\frac{\Gamma' \longrightarrow \Delta', \exists x B(x), B(t)}{\Gamma' \longrightarrow \Delta', \exists x B(x)}$$

The term t is used in this case as well.

Clearly, if we are done are finitely many steps, then the last of P is $\Pi, \Gamma \longrightarrow \Delta$, and we are done. What if we do not halt after finitely many steps? We will show that the hypothesis of the Completeness Theorem fails, i.e.

$$\begin{aligned} \forall \mathbb{S} \quad &\not\models \quad \Gamma \longrightarrow \Delta \text{ , or in the other formulation,} \\ T \quad &\not\models \quad \Gamma \longrightarrow \Delta \end{aligned}$$

However, this construction will have to wait until next time.