Math 260A — Mathematical Logic — Scribe Notes
UCSD — Winter Quarter 2012
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1 LK Soundness

1.1 Definitions

Let

$$S = A_1, ..., A_k \longrightarrow B_1, ..., B_l$$

Let

$$\forall S = \forall x_1..\forall x_j(A_1 \wedge ... \wedge A_k \longrightarrow B_1 \vee ... \vee B_l)$$

We can infer $\forall S$ from S.

$$\cfrac{\cfrac{\cfrac{A_1, ..., A_k \longrightarrow B_1, ..., B_l}{A_1 \wedge ... \wedge A_k \longrightarrow B_1, ..., B_l}\ \wedge\text{:left, exchanges}}{A_1 \wedge ... \wedge A_k \longrightarrow B_1 \vee ... \vee B_l}\ \vee\text{:right, exchanges}}{\forall x_1..\forall x_j(A_1 \wedge ... \wedge A_k \longrightarrow B_1 \vee ... \vee B_l)}\ \forall\text{:left}$$

The final line constitutes a universal closure.

Soundness #1 If $LK \vdash \Gamma \longrightarrow \Delta$, then $\Gamma \longrightarrow \Delta$ is valid. In other words, if $LK \vdash S$, $\forall S$ is valid.

Soundness #2 If $LK_e \vdash S$, $\forall S$ is valid.

Soundness #3 Let $LK_{\mathbb{S}}$ be the following proof system. An $LK_{\mathbb{S}}$ proof of $\Gamma \longrightarrow \Delta$ is a sequence ofd sequents such that each sequent is in $\mathbb{S}$, or is an initial axiom $A \longrightarrow A$, or is inferred from earlier sequents.

If $LK_{\mathbb{S}} \vdash S$, then $\forall \mathbb{S} \models S$ where $\forall \mathbb{S} = \{\forall S' : S' \in \mathbb{S}\}$

Proof of Soundness Assume $\mathcal{M} \models \forall \mathbb{S}$. If S is a sequent in the proof then forall σ, it is the case that $\mathcal{M} \models S$. Show by induction on the number of lines in the proof.

2 LK Completeness

Completeness #1 If $\Gamma \longrightarrow \Delta$ is valid, then there is an LK proof of $\Gamma \longrightarrow \Delta$.

Completeness #2 If $\Gamma \longrightarrow \Delta$ is valid and involves equality, then there is an LK_e proof of $\Gamma \longrightarrow \Delta$.

Completeness #3 If $\mathbb{S}$ is a set of sequents and $\forall \mathbb{S} \models \Gamma \longrightarrow \Delta$, then (a) there is an LK proof of $\Pi, \Gamma \longrightarrow \Delta$ (where Π is a finite subset of $\forall \mathbb{S}$), and (b) there is an $LK_{\mathbb{S}}$ proof of $\Gamma \longrightarrow \Delta$ (where if $\Gamma \longrightarrow \Delta$ or $\mathbb{S}$ uses equality, then $\mathbb{S}$ includes the equality axioms).

Proof that a proves b If $\Pi \longrightarrow \Lambda$, where $\Lambda \in \mathbb{S}$ then $\forall(\Pi \rightarrow \Lambda), \Pi \longrightarrow \Lambda$. Each $c \in \Pi$ is of the form $\forall S$, where $S \in \mathbb{S}$. And there is an LK proof from S to $\forall S$ and vice versa.

Proof of Completeness #3 (a) Restatement:

Let $\mathbb{S}$ be as above. Assume $\forall \mathbb{S} \models \Gamma \longrightarrow \Delta$. Then there is a cut-free LK proof of $\Pi, \Gamma \longrightarrow Delta$ for some finite subset Π of $\forall \mathbb{S}$.

The proof is... to be continued on Wednesday.

(I've omitted the discussion about cut free proofs and equality axioms. If this results in public outcry on the blog, I'll amend the notes.)