Math 260B — Mathematical Logic — Scribe Notes
UCSD — Spring Quarter 2012
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Last time, we proved that there is a many-one reduction from H to V_L, where $V_L := \{\varphi : \varphi \text{ is a valid L-sentence}\}$.

Corollary 1. *V_L is r.e. complete (many-one complete for the r.e. sets).*

Proof. For all r.e. sets R, there is a many-one reduction from R to V_L, since H is r.e. complete and many-one reducibility is transitive.

Also, V_L is r.e., since V_L is the range of a recursive function f defined as follows:

$$f(w) = \begin{cases} \varphi & \text{if } w \text{ codes a valid } L\text{-proof with last one } \varphi; \\ \forall x(x = x) & \text{otherwise.} \end{cases}$$

□

1 Validity in Finite Models

Let $V_L^{\text{finite}} := \{\varphi : \varphi \text{ is an } L\text{-sentence valid in all finite models}\}$.

Theorem 1. *For the language L before, V_L^{finite} is co-r.e. complete.*

Lemma 1. *V_L^{finite} is co-r.e.*

Proof. Since L is finite, we can enumerate all of finitely many structures of each finite cardinality $n \geq 1$.

Algorithm to semi-decides $\overline{V_L^{\text{finite}}}$.
For $i =, 1, 2, 3, \ldots$
Enumerate all L-structures of size i
If φ is false in any one, then accept
End loop.

□

Proof of the theorem

Modify Friday's Proof to obtain a many-one reduction from $\overline{H}$ to V_L^{finite}. Our new language is $L = \{0, S, T, q, \tau, \sigma\}$, where T ("top") is a new constant symbol.

ψ is modified as follows:

1. $\forall x S(x) \neq 0$;
2. $\forall x \forall y (S(x) = S(y) \to x = y \lor x = T \lor y = T)$
   i.e., $\forall x \forall y (S(x) = S(y) \land x \neq T \land x \neq T \to x = y)$;
3. $\forall x (x \neq 0 \to \exists y (S(y) = x))$;
4. For any $k \geq 0$, $\forall x (S^k(x) = x \to S^k(x) = T)$

We call this ψ^T. Note that T is the successor of two elements: T itself and another element.

No changes are needed to χ_1, χ_2, or χ_4.

For χ_3, the change is as follows. For $(q_2, b), (0, R, q_3)$, for example,

$$\begin{aligned}&\forall x \forall y[\ (q(x, S(S(0))) \land \tau(x, y) \land \neg\sigma(x, y) \land \underline{x \neq T}) \quad \to \\ &(q(S(x), S(S(0))) \land \tau(S(x), S(y)) \land \sigma(S(x), y))\ ]\end{aligned}$$

We call this χ_3^T.

Lemma 2. *$\psi^T \land \chi_1 \land \chi_2 \land \chi_3^T \land \chi_4 \to \neg(q(T, S(0)))$ if and only if M does not halt when χ_3^T is the axioms specific to M.*

Notice that $f : \ulcorner M \urcorner \mapsto \ulcorner \varphi_M \urcorner$ is recursive, where φ_M is $\psi^T \land \chi_1 \land \chi_2 \land \chi_3^T \land \chi_4 \to \neg(q(T, S(0)))$. The lemma says $\ulcorner M \urcorner \in \overline{H} \iff \ulcorner \varphi_M \urcorner \in \{\ulcorner \varphi \urcorner :$ φ is an L-sentence valid in all finite models$\}$.

Proof. By construction. □

Corollary 2. *There is no abstract proof system for the set of sentences valid in finite models of L.*

Thus, there is no recursive function whose range is the set of formulas valid in all finite models.

2 Robinson's Theory Q

The language of Robinson's Theory Q contains 0, S, $+$, $\cdot$, $\leq$. The axioms are as follows:

1. $(\forall x)(Sx \neq 0)$;
2. $(\forall x)(\forall y)(Sx = Sy \to x = y)$;
3. $(\forall x)(x \neq 0 \to (\exists y)(Sy = x))$;
4. $(\forall x)(x + 0 = x)$;
5. $(\forall x)(\forall y)(x + Sy = S(x + y))$;
6. $(\forall x)(x \cdot 0 = 0)$;
7. $(\forall x)(x \cdot Sy = x \cdot y + x)$;
8. $(\forall x)(\forall y)(x \leq y \leftrightarrow (\exists z)(x + z = y))$.

Theorem 2. *$Q \nvdash (\forall x)(Sx \neq x)$.*

Proof. A model $\mathcal{M}$ of $Q \cup \{(\exists x)(Sx = x)\}$ is the following:

(a) $|\mathcal{M}| = \mathbb{N} \cup \{\infty\}$;

(b) $S(\infty) = \infty$;

(c) for all $x \in \mathbb{N} \cup \{\infty\} \setminus \{0\}$,

 (c-1) $x + \infty = \infty + x = \infty$;
 (c-2) $x \cdot \infty = \infty \cdot x = \infty$;

(d) $0 + \infty = \infty + 0 = \infty$;

(e) $0 \cdot \infty = \infty \cdot 0 = 0$.

□