PRIMITIVE RECURSIVE FUNCTIONS

0-ary, nullary	$z(\ldots)$
Successor	$s(n)$
Identity	$\prod_i^n(x_1, \ldots, x_n) = x_i$
Composition	$f(x_1, \ldots, x_n) = g(h_i(x_1, \ldots, x_n), \ldots, h_k(x_1, \ldots, x_n))$

1. Addition

2. Multiplication

$0 \cdot n = 0$ $\quad g(0, n) = 0$

$(m+1) \cdot n = m \cdot n + n$ $\quad g(m+1, n) = g(m, n) + n \quad = f(g(m, n), n)$

- Want this form: $h(m, g(m,n), n) = f(\underbrace{\prod_2^3(m, g(m.n), n)}_{g(m,n)}, \underbrace{\prod_3^3(m, g(m.n), n)}_{n})$

* **Define**: $h(a.b.c) = f(\prod_2^3(a.b.c), \prod_3^3(a, b, c))$

3. Exponentiation

$k(x, y) = x^y$ $\quad x^0 = 1$

$x^{m+1} = x^m \cdot x = g(x^m, x)$

4. Iterated Exponentiation

$x \Uparrow y = \left. x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}} \right\} y$ $\quad x \Uparrow 0 = 1$

$x \Uparrow (m+1) = x^{(x \Uparrow m)}$

BOOTSTRAPPING

Definition: Where R is a relation on k-tuples: if $R \subseteq \mathbb{N}^{\boldsymbol{k}}$ the characteristic function of R is:

$$x_R(\vec{x}) = \overrightarrow{\begin{cases} 1 & \text{if } \vec{x} \in R \\ 0 & \text{otherwise} \end{cases}}$$

R is primitive recursive iff its characteristic function is.

5. The relation $x > 0$ is primitive recursive

$f(0) = 0$

$f(n + 1) = 1 = s(0) = s(z(\ldots))$

6. The predecessor function is primitive recursive

$p(0) = 0$

$p(n+1) = n = \Pi_1^3(n, P(n))$

7. The restricted subtraction function is primitive recursive

$$x \dot{-} y = \begin{cases} x - y & \text{if } x - y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$x \dot{-} 0 = x$

$x \dot{-} (m+1) = p(x \dot{-} m)$

Theorem: If R is primitive recursive, and f and g are primitive recursive, then:

$$h(\vec{x}) = \begin{cases} f(\vec{x}) & \text{if } \mathrm{R}(\vec{x}) \\ g(\vec{x}) & \text{otherwise} \end{cases}$$

Proof: $h(\vec{x}) = f(\vec{x}) \cdot \chi_R(\vec{x}) + g(\vec{x}) \cdot (1 \dot{-} \chi_R(\vec{x}))$

8. If R is primitive recursive, so is the negation/complement of R

$\chi_{\bar{R}}(\vec{x}) = 1 \dot{-} \chi_R(\vec{x})$

9. If R and Q are both primitive recursive, then so are $R \cap Q$ and $R \cup Q$

$\chi_{R \cap Q} = \chi_R \cdot \chi_Q$

$R \cup Q = \overline{\bar{R} \cap \bar{Q}}$

10. R(x,y) def. "x=y" is primitive recursive

$\chi_{x=y} = \neg((x \dot{-} y) + (y \dot{-} x) > 0)$

"NOW WE HAVE SOME FUN STUFF"

11. Limited Product

$$f(z,\vec{y}\,) = \prod_{i=0}^{z-1} g(i,\vec{y}\,)$$

Theorem: If *g* is primitive recursive, so is *f*.

Proof: $f(0,\vec{y}\,) = 1$

$f(m+1,\vec{y}\,) = g(m,\vec{y}\,) \cdot f(m,\vec{y}\,)$

12. Limited Sum

$$f(z,\vec{y}\,) = \sum_{i=0}^{z-1} g(i,\vec{y}\,)$$

$f(0,\vec{y}\,) = 0$

$f(m+1,\vec{y}\,) = f(m,\vec{y}\,) + g(m,\vec{y}\,)$

13. If R(x, $\vec{y}$)is a primitive recursive relation then so are the relations obtained from R by bounded quantification

$\mathrm{Q}(z,\vec{y}) \coloneqq (\forall x \leq z) R(x,\vec{y})$

$\mathrm{S}(z,\vec{y}) \coloneqq (\exists x \leq z) R(x,\vec{y})$

Proof:

$$\chi_Q(z,\vec{y}) = \prod_{i=0}^{z} \chi_R(\chi,\vec{y})$$

14. The divides relation, $\mathrm{R}(x,y)$*= x divides y, is primitive recursive*

$\mathrm{R}(x,y) \Leftrightarrow x|y$

$\mathrm{R}(x,y) \Leftrightarrow (\exists z \leq y)(x \cdot z = y)$

15. The prime relation $\mathrm{Prime}(x)$ *is primitive recursive*

$\mathrm{Prime}(x) \Leftrightarrow (\forall z < x)(z > 1 \ \rightarrow \neg(z|x)\,)$

$\Leftrightarrow (\forall z < x)(\neg(z > 1) \ \vee \ \neg(z|x)\,)$

16. Bounded Minimization

Let $\mathrm{R}(x, \vec{y})$ be primitive recursive.

Define: $f(z, \vec{y}) = (\mu x \leq z)\mathrm{R}(x, \vec{y})$; $\mu = least\ x\ bounded\ by\ value\ z\ s.t.\ R(x, \vec{y})\ holds$

$$= \begin{cases} \text{least } x \leq z \text{ s.t. } \mathrm{R}(x, \vec{y}) \\ z + 1 \text{ if no such } x \text{ exists} \end{cases}$$

$f(z, \vec{y})$ is primitive recursive:

$$f(0, \vec{y}) = \begin{cases} 0 & \text{if } \mathrm{R}(0, \vec{y}) \\ 1 & \text{otherwise} \end{cases}$$

$$f(m + 1, \vec{y}) = \begin{cases} f(m, \vec{y}) & \text{if } f(m, \vec{y}) \leq m \\ m + 1 & \text{if } f(m, \vec{y}) = m + 1 \wedge \mathrm{R}(m + 1, \vec{y}) \\ m + 2 & \text{otherwise} \end{cases}$$

17. The next prime function is primitive recursive

$\text{Next Prime}(x) = (\mu y \leq 2x + 2)(y > x \text{ and } y \text{ is prime})$

18. The ith prime function is primitive recursive

$$i\text{th Prime}(0) = 2$$
$$i\text{th Prime}(n + 1) = \text{Next Prime}(i\text{th Prime}(n))$$

NEXT: SEQUENCE CODING