Math 260A — Mathematical Logic — Scribe Notes
UCSD — Winter Quarter 2012
Instructor: Sam Buss

Notes by: Bob Chen
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1 A Characterization of Recursively Enumerable Sets

Definition 1.1. Let $R(x, \vec{w})$ be a relation $R \subset \Sigma^\star \times (\Sigma^\star)^k$. Define $Q(\vec{w})$ by $(\exists x \in \Sigma^\star)R(x, \vec{w})$. We call Q a *projection* of R.

Theorem 1.2. *Q is a projection of a recursive set if and only if Q is r.e.*

Proof. Suppose Q is a projection of an r.e. set R. Let N be a TM that semidecides R. For each $i = 0, 1, 2, \ldots$, run $N(x, \vec{w})$ for i steps on every $x \in \Sigma^\star$ with length at most i. If $\vec{w}$ has a witness for R, N will find it and accept. This semidecides Q, so Q is r.e. (an easy change can give us an enumerator).

Now suppose Q is r.e. Then Q is the range of a recursive function f (if Q is empty then we're done). So $\vec{w} \in Q$ if and only if $\exists x(f(x) = \vec{w})$, the insides of which is a recursive predicate. □

2 Many-One Completeness

Recall 2.1. The Halting problem H defined by

$$H := \{\ulcorner M \urcorner \mid M \text{ halts on input } \epsilon\}$$

is r.e.

Claim 2.2. *"H is the hardest r.e. set."*

Recall 2.3. We say f is a many-one reduction from R to Q if $f : R \to Q$ is a recursive function such that for all w, $w \in R$ if and only if $f(w) \in Q$. In this case we write $R \leq_m Q$.

Theorem 2.4. *Suppose R is r.e. Then there is a many-one reduction from R to H.*

Proof. R is r.e. and hence the domain of a partial recursive function computed by some M_R. Define $f(w) := \ulcorner M_w \urcorner$ where M_w is the TM that runs $M_R(w)$. Then M_w halts if and only if $w \in R$. □

Theorem 2.5. *Q is r.e. and $R \leq_m Q$ implies that R is r.e.*

Definition 2.6. R is *many-one complete* for the r.e. sets if

- R is r.e., and
- every r.e. set is many-one reducible to R.

Theorem 2.7. *H is many-one complete for the r.e. sets.*

Example 2.8. (Hilbert's 10th Problem.) There is a particular polynomial p with integer coefficients such that the solutions

$$M_p := \{n \in \mathbb{N} \mid \exists m_1, \ldots m_{24} p(n, m_1, \ldots, m_{24}) = 0\}$$

to the Diophantine equation is r.e.-complete.

3 Primitive Recursive Functions and Predicates

Definition 3.1. Given a k-ary function $g(\vec{x})$ and a $(k+2)$-ary function $h(n, y, \vec{x})$ we can *define f from g, h by primitive recursion* by setting

$$\begin{cases} f(0, \vec{x}) = g(\vec{x}) \\ f(n+1, \vec{x}) = h(n, f(n, \vec{x}), \vec{x}). \end{cases}$$

Definition 3.2. The set of primitive recursive functions is the smallest set containing

- the constantly 0 function,
- the successor function,
- the projection functions,

and is closed under function composition and primitive recursion.

Example 3.3. $f(x, y) = x + y$ is primitive recursive. Use primitive recursion with $g = \pi_1^1$ and $h = S \circ \pi_2^3$.

Example 3.4. Multiplication and exponentiation are primitive recursive.