Math 260A — Mathematical Logic — Scribe Notes
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Instructor: Sam Buss

Notes by: Stephen R. Foster
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1 Alternative Proof of Halting Problem

"Quine" theorem restated: For a partial recursive function $f(x, \vec{w})$, there exists a Turing Machine M such that $M(\vec{w}) = f(\ulcorner M \urcorner, \vec{w})$.

Using this, we can obtain another proof of the Halting Problem.

Suppose some machine M_0 solves the halting problem. We can construct M such that $M(\ulcorner N \urcorner) \downarrow \leftrightarrow N() \uparrow$.

By the "Quine" theorem: There is some M_2 such that $M_2() = M(\ulcorner M_2 \urcorner)$.

Thus $M_2() \downarrow \leftrightarrow M(\ulcorner M_2 \urcorner) \downarrow \leftrightarrow M_2() \uparrow$. This is a contradiction.

2 Second Recursion Theorem / Kleene's Fix Point Theorem

Let f be a (total) recursive function. There exists a Turing Machine M such that $U(\ulcorner M \urcorner, w) = U(f(\ulcorner M \urcorner, w)$ for all $w \in \Sigma^*$. That is, M computes the same partial function as a Turing Machine with Gödel number $f(\ulcorner M \urcorner)$.

(By the way, why do we need to specify "partial"? Consider $f(x) = \ulcorner M_0 \urcorner$, where M_0 is a Turing Machine that loops forever.)

2.1 Notational aside

If Turing Machine M has Gödel number $e = \ulcorner M \urcorner \in \Sigma^*$, we write that $\{e\} = \{\ulcorner M \urcorner\}$ is the partial function computed by M.

The function $\{e\}$ has type $\Sigma^* \to \Sigma^*$.

2.2 The Proof continued

Let $g(x, w)$ be such that $g(\ulcorner N \urcorner, w) = N'(w)$, where $\ulcorner N' \urcorner = f(\ulcorner N \urcorner)$. Or, using the alernative notation: $g(e, w) = \{f(e)\}(w)$.

The algorithm for defining g would simply be: 1) compute $f(e)$, then 2) run $U(f(e), w)$.

We know there's some $M_2(w) = g(\ulcorner M_2 \urcorner, w)$. So $U(\ulcorner M_2 \urcorner, w) = M_2(w) = g(\ulcorner M_2 \urcorner, w) = U(f(\ulcorner M_2 \urcorner), w)$. This completes the proof.

3 Rice's Theorem

Let C be a set of partial recursive functions. Let $C\#$ be Gödel numbers $\{e : \{e\} \in C\}$. Suppose $C\#$ is non-empty and $C\#$ is not the set of natural numbers. Then $C\#$ is not decidable.

(This generalizes the halting problem by suggesting that any non-trivial property of what a Turing Machine computes is not decidable.)

To prove this, let M_0 be such that $\{\ulcorner M_0 \urcorner\} \in C$ and $\{\ulcorner M_1 \urcorner\} \notin C$. Assume $C\#$ is decidable. Let $g(\ulcorner M \urcorner, w)$ be the function that returns $M_0(w)$ if $M \in C$ and returns $M_1(w)$ if $M \notin C$. Since $C\#$ is decidable, g is partial recursive.

By the "Quine" theorem: Let $M_2(w) = g(\ulcorner M_2 \urcorner, w)$. If $\ulcorner M_2 \urcorner \in C\#$, then $g(\ulcorner M \urcorner, w) = M(w)$. Thus $M_2(w) = g(\ulcorner M \urcorner, w) = M_1(w)$, and $\ulcorner M_2 \urcorner \in C\#$. This is a contradiction. A similar argument can be used if $\ulcorner M_2 \urcorner \notin C\#$.