**Math 20F - Homeworks 9 & 10 - Selected answers**

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**Section 6.1, Problem 19.** We did almost exactly this problem as a theorem proved in class.

**Section 6.1, Problem 22.** Since $\mathbf{u}_j^T\mathbf{u}_i = \delta_{i,j}$, we have

$$A\mathbf{u}_i \;=\; \sum_{j=1}^{n} c_j\mathbf{u}_j\mathbf{u}_j^T\mathbf{u}_i \;=\; \sum_{j=1}^{n}\delta_{i,j}c_j\mathbf{u}_j \;=\; c_i\mathbf{u}_i.$$

Hence $\mathbf{u}_i$ is an eigenvector for the eigenvalue $c_i$.

**Section 6.3, Problem 1(a).** We did this as an example in class on Wednesday.

**Section 6.3, Problem 1(c).** $\det(A-\lambda I) = (2-\lambda)(-4-\lambda)+8 = 2\lambda+\lambda^2 = (2+\lambda)(\lambda)$. The roots of the characteristic polynomial are $\lambda_1 = 1$ and $\lambda_2 = -2$: these are the eigenvalues of $A$.

Solving $A\mathbf{x} = 0$ for a nontrivial $\mathbf{x}$, we find that $\mathbf{x}_1 = (4,1)^T$ is a eigenvector corresponding to the eigenvalue $\lambda_1 = 0$.

Solving $(A+2I)\mathbf{x} = 0$, we find that $\mathbf{x}_2 = (2,1)^T$ is an eigenvector corresponding to the eigen value $\lambda_2 = -2$.

The eigenvalues are distinct, hence $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. Therefore, $A$ is diagonalizable in the following form:

$$\begin{aligned} A &= \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & -1 \\ -1/2 & 2 \end{pmatrix} \end{aligned}$$

**Section 6.3, Problem 3(c).** The sixth power of $A$ is equal to

$$A^6 = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}^6 \cdot \begin{pmatrix} 1/2 & -1 \\ -1/2 & 2 \end{pmatrix}$$

This can be multiplied out by hand if desired – start by computing the sixth power of the diagonal matrix. This particular case is rather easy since there is only one non-zero eigenvariable. You will find that

$$A^6 = \begin{pmatrix} -64 & 256 \\ -32 & 128 \end{pmatrix}.$$

**Section 6.3, Problem 4.** The idea behind these problems is as follows. First diagonalize the matrix $A$ as $A = SDS^{-1}$. If the eigenvalues are non-negative, the diagonal matrix $D$ has non-negative entries along the diagonal. A matrix $E$ such that $E^2 = D$ can be formed by letting $E$ be the diagonal matrix whose entries are the square roots of the entries of $D$. Then, letting $B = SES^{-1}$, we have $B^2 = A$.

**Section 6.3, Problem 8(a), 9.** I neglected to define "defective" in class Wednesday (although I intended to). Therefore, as I promised, this term will not appear on the final exam. An $n \times n$ matrix that does not have $n$ linearly independent eigenvectors is called *defective*. That is to say, a matrix is diagonalizable if and only if it is not defective.

**Section 6.3, Problem 9.** If $A$ has one eigenvalue (call it $\lambda_1$) of multiplicity 3, then the other eigenvalue (call it $\lambda_2$) has multiplicity 1. Now, there is an eigenvector $\mathbf{x}_2$ for $\lambda_2$ of course. Furthermore, since $rank(A - \lambda_1 I)$ is equal to 1, then the null space of $A - \lambda_1 I$ has dimension 3; therefore, there are three linearly independent eigenvectors for $\lambda_1$. Further, since $\lambda_1 \neq \lambda_2$ (they are unequal, since otherwise, the eigenvalue would have multiplicity four!), $\mathbf{x}_2$ is not in the eigenspace of $\lambda_1$. Thus, the four eigenvectors are linearly independent, so $A$ is diagonalizable and $A$ is not defective.