Math 20F - Linear Algebra

Midterm Examination #2 – ANSWERS

Spring, May 23, 2003 — Instructor: Sam Buss

Write your name or initials on every page before beginning the exam.

You have 50 minutes. There are 7 problems. You may not use notes, textbooks, calculators, or other materials during this exam. You must show your work in order to get credit. Good luck!

Name:

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1.

a. State the definition of "the range of A, $R(A)$".

ANSWER: $\{\mathbf{b} : A\mathbf{x} = \mathbf{b}$ *is consistent*$\}$

b. State the definition of "$\mathbf{u}_1, \ldots, \mathbf{u}_k$ is a basis for U".

ANSWER: $\mathbf{u}_1, \ldots, \mathbf{u}_k$ *are linearly independent and span* U.

c. State the definition of "$f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation".

ANSWER: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and all* $\alpha, \beta \in \mathbb{R}$,

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}).$$

d. State the definition of "A represents the linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$".

ANSWER: For all $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x} = f(\mathbf{x})$.

2. Suppose that U is a subspace of $\mathbb{R}^n$ and $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is a spanning set for U. For each of the following items, tell whether it *MUST* be true. (Write "Yes" if it must be true; write "No" if it is not necessarily true.

a. $k \leqslant n$. *NO*

b. $k \geqslant n$. *NO*

c. The dimension of U is at most k. *YES*

d. The dimension of U is at least k. *NO*

e. The dimension of $U^\perp$ is at least k. *NO*

f. The dimension of $U^\perp$ is at least $n - k$. *YES*

3. Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{u}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{u}_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$.

Let $U = Span(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5)$.

a. Are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ linearly independent?

b. Find a subset of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$ that is a basis for U.

ANSWER: a. The five vectors are not linearly independent. One way to know this is that they lie in a 4-dimensional space.

b. Suggested method for solving this is to use row operations on the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

This will show that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ *are a basis for the column space of the matrix, and hence a basis for* U.

(Other answers are possible, but require justification.)

4. Let $A = \begin{pmatrix} 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & -1 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

a. Calculate the rank of A. *ANSWER: 3*

b. Calculate the dimension of $R(A)$. *ANSWER: 3*

c. Calculate the dimension of $N(A)$. *ANSWER: 2*

d. Calculate the dimension of $R(A^T)$. *ANSWER: 3*

e. Calculate the dimension of $N(A^T)$. *ANSWER: 1*

f. Find a basis for the row space of A. Explain how you can be sure it is a basis. *ANSWER:* $(1,2,2,1,1)^T$, $(0,1,1,0,0)^T$, *and* $(0,0,1,1,0)^T$, *is a basis. This is a basis since they are the three non-zero rows in the row echelon form of* A, *since the row space of* A*'s REF is the same as the rowspace of* A.

(Other answers are possible, it depends on the order in which you do the row operations.) One possible result for the row echelon form of A *is:*

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined as follows. To calculate $g(\mathbf{x})$, you first rotate $\mathbf{x}$ counterclockwise 45 degrees around the origin, and then you reflect the resulting vector across the y-axis (the x_2-axis) to obtain the value of $g(\mathbf{x})$. Find a matrix A that represents g.

ANSWER: $A = \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$.

6. Find the linear function $y = f(x)$ that best approximates the data

x	-2	-1	0	2
y	2	1	0	2

in the least squares sense.

ANSWER: $f(x) = \frac{44}{35} + \frac{1}{35}x$.

7. Let $\mathbf{u} = (1,\, -1,\, 2)^T$. The subspace U of $\mathbb{R}^3$ is defined by

$$U \;=\; \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{u} = 0\}.$$

a. Find a basis for $U^\perp$.

b. Find a basis for U.

c. Let $\mathbf{x} = (1,\, 0,\, 0)^T$. Find the vector $\mathbf{p} \in U$ which the closest point in U to $\mathbf{x}$.

d. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by letting $g(\mathbf{x})$ equal the projection of $\mathbf{x}$ onto the subspace $U^\perp$. Find the matrix that represents the function $g(\mathbf{x})$.

ANSWER: The key to this problem is to note that $U = Span(\mathbf{u})^\perp$. *Thus,* $U^\perp = Span(\mathbf{u})$.

a. $\mathbf{u}$ *(by itself) is a a basis for* $U^\perp$.

b. U *is the nullspace of the* 3×1 *matrix* $(1, -1, 2)$. *Since this is already in rref form, it can be solved immediately. Then, a basis for* U *is* $(-2, 0, 1)^T, (1, 1, 0)^T$.

c. First project $\mathbf{x}$ *onto* $U^\perp$*, to get*

$$\mathbf{q} \;=\; \mathbf{x} - \frac{(\mathbf{x} \cdot \mathbf{u})\mathbf{u}}{||\mathbf{u}||^2} \;=\; (1/6, -1/6, 1/3)^T.$$

Then, the answer is $\mathbf{p} = \mathbf{x} - \mathbf{q} = (5/6, 1/6, -1/3)^T$.

d. The matrix is

$$\frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}} \;=\; \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}$$