Math 20F - Linear Algebra

Midterm Examination #1

Spring, April 28, 2003 — Instructor: Sam Buss

Write your name or initials on every page before beginning the exam.

You have 50 minutes. There are six problems. You may not use notes, textbooks, calculators, or other materials during this exam. You must show your work in order to get credit. Good luck!

Name:

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1. Let A be an $m \times n$ matrix.

a. State the definition of "the nullspace of A, $N(A)$".

ANSWER: $N(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$.

b. State the definition of "A is skew-symmetric".

ANSWER: $A = -A^T$.

2. Suppose that a 3×3 matrix A is row equavalent to

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

(However, you do not know the matrix A itself.) For the following questions, either give an answer, or state "insufficient information" if you do not have enough information to answer the question.

a. What is the value of $\det(A)$? *ANSWER: 0.*

b. Is A singular? *ANSWER: Yes*

c. How many solutions does the equation $A\mathbf{x} = 0$ have? *ANSWER: Infinitely many.*

d. Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution?
ANSWER: Insufficient information.

3. Let O be the 2×2 matrix containing all zeros. Do there exist 2×2 matrices A and B such that $A \neq O$ and $B \neq O$, and such that $AB = O$? Either explain why there cannot exist such matrices, **or** give an example of matrices A and B with these properties.

ANSWER. There are many possible answers. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

4. Compute the inverse of A, where $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Calculate the determinant of A.

ANSWER: $\det(A) = 1$ *and* $A^{-1} = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

5. *Let S be the subset of $\mathbb{R}^2$ defined by $S = \{(x_1,\, x_2)^T : x_1 \geqslant 0\}$. Is S a subspace of $\mathbb{R}^2$? Prove your answer.*

ANSWER: S is not a subspace. Although it is closed under addition, it is not closed under scalar multiplication.

For example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S$, *but* $(-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \notin S$.

6. *Let $\mathbf{x} = (8,\, 4,\, -1,\, -7)^T$. Let $\mathbf{v}_1 = (1,\, 0,\, 0,\, 1)^T$, $\mathbf{v}_2 = (3,\, 2,\, 1,\, 0)^T$, and $\mathbf{v}_3 = (0,\, 1,\, 2,\, 3)^T$. Is $\mathbf{x} \in Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$? If not, explain why not. If so, express $\mathbf{x}$ explicitly as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.*

ANSWER: The method to solve this problem is as follows. Set $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix}$ *and solve*

$$A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \mathbf{x}.$$

This equation has a (unique) solution $\alpha_1 = -1$, $\alpha_2 = 3$, and $\alpha_3 = -2$. Therefore, $\mathbf{x} \in Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, and

$$\mathbf{x} = (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + (-2)\mathbf{v}_3.$$