Math 160B - Winter 2022 - Quiz 1 - Upload by 11:00am, Wednesday, January 12

Start Time: **Your name:** Answer Key

Stop Time: **Integrity signature:**

Time limit 15 minutes, not counting download and upload. Please add explanation if over 17 minutes.

1. Let $\mathcal{R}'$ be the structure $\mathcal{R}' = (\mathbb{R}, 0, 1, +, \cdot, <)$. Let T be the theory $\mathrm{Th}\, R'$. Prove T has a non-archimedean model.

A non-archimedean model is a structure $\mathcal{A}$ that has an object $a \in |\mathcal{A}|$ such that (a) a is positive, and (b) a is greater than any fixed integer $n \in \mathbb{N}$. Note that $\mathcal{R}'$ is archimedean since every positive $x \in \mathbb{R}$ is less than or equal to n where $n = \lceil x \rceil$. More precisely, condition (a) means that $\mathcal{A} \vDash a > 0$; and condition (b) means that, for all $n \in \mathbb{N}$, $\mathcal{A} \vDash \underline{n} < a$ where $\underline{n}$ is the term $1 + 1 + \cdots + 1$ representing n.

Let T' be $T \cup \{\underline{n} < c : n \in \mathbb{N}\}$ where c is a new constant symbol.

Claim: T' is finitely satisfiable.

Pf of claim: Let Δ be a finite subset of T'. Then $\Delta \subseteq T \cup \{\underline{n} < c : n \le n_0\}$ for some $n_0 \in \mathbb{N}$. Δ is satisfied by $\mathcal{R}'$ expanded to the language $\langle 0, 1, +, \cdot, <, c\rangle$ by letting the interpretation of c be $n+1$. That proves the claim.

Therefore, by Compactness, T' has a model $\mathfrak{A}$. This model must be non-archimedean, since $c^{\mathfrak{A}}$ will satisfy $\mathfrak{A} \vDash 0 < c^{\mathfrak{A}}$ and $\mathfrak{A} \vDash \underline{n} < c^{\mathfrak{A}}$ for all $n \in \mathbb{N}$.