Second Incompleteness theorem

$\mathrm{PRF}_T(w, \ulcorner A \urcorner)$ ← representable / decidable

$\mathrm{THM}_T(\ulcorner A \urcorner)$ - $\exists w\, \mathrm{PRF}_T(w, \ulcorner A \urcorner)$ ← not decidable hence not representable

Diagonal Formula

$$R \vdash D \leftrightarrow \neg \mathrm{THM}_T(\ulcorner D \urcorner)$$

Show $T \nvdash D$ (T - consistent, axiomatizable $T \supseteq R$)

$T \nvdash \neg D$ (T is also ω-consistent)

Jumping-off point for 2nd Incompleteness theorem

If T is axiomatizable and $T \supseteq R$,

Then if T is consistent then $T \nvdash D$

i.e. D is true

Def'n CON_T is the sentence $\neg THM_T(\underline{\ulcorner 0 = 1 \urcorner})$

Note $R \vdash 0 \neq 1$

Goal $T \nvdash CON_T$ (2nd incompleteness theorem)

Need more assumptions Hilbert-Bernays-Löb conditions.

For all sentences A, B:

HBL-1 If $T \vdash A$ then $T \vdash THM_T(\underline{\ulcorner A \urcorner})$

HBL-2 $T \vdash THM_T(\underline{\ulcorner A \urcorner}) \to THM_T(\underline{\ulcorner THM_T(\underline{\ulcorner A \urcorner}) \urcorner})$

HBL-3 $T \vdash THM_T(\underline{\ulcorner A \urcorner}) \wedge THM_T(\underline{\ulcorner A \to B \urcorner})$
$\to THM_T(\underline{\ulcorner B \urcorner})$

(Formalized Modus Ponens)

PA - satisfies HBL 1 - 3. ✓

<u>Lemma</u> Suppose T satisfies HBL 1-3 & $T \supseteq R$ & is ~~cons~~ axiomatizable

(a) $T \vdash \neg \mathrm{CON}_T \to \mathrm{THM}_T(\ulcorner \underline{A} \urcorner)$

(b) $T \vdash \mathrm{THM}_T(\ulcorner \underline{A} \urcorner) \wedge \mathrm{THM}_T(\ulcorner \underline{\neg A} \urcorner) \to \neg \mathrm{CON}_T$.

(a) restated: $T \vdash \neg \mathrm{THM}_T(\ulcorner \underline{A} \urcorner) \to \mathrm{CON}_T$

<u>Pf</u> (a) Use $T \vdash 0=1 \to A$ (Since $T \vdash 0 \neq 1$)

Thus by HBL-1 $T \vdash \mathrm{THM}_T(\ulcorner \underline{0=1 \to A} \urcorner)$

by HBL-2 $T \vdash \mathrm{THM}_T(\ulcorner \underline{0=1} \urcorner) \wedge \mathrm{THM}_T(\ulcorner \underline{0=1 \to A} \urcorner)$
$\to \mathrm{THM}_T(\ulcorner \underline{A} \urcorner)$.

(a) follows from these

(b) Use $T \vdash A \to \neg A \to 0=1$ so $T \vdash \mathrm{THM}_T(\ulcorner \underline{A \to \neg A \to 0=1} \urcorner)$ by HBL-1

HBL-3: $\mathrm{THM}_T(\ulcorner \underline{A} \urcorner) \to \mathrm{THM}_T(\ulcorner \underline{A \to \neg A \to 0=1} \urcorner) \to \mathrm{THM}_T(\ulcorner \underline{\neg A \to 0=1} \urcorner)$

and $\mathrm{THM}_T(\ulcorner \neg A \urcorner) \to \mathrm{THM}_T(\ulcorner \underline{\neg A \to 0=1} \urcorner) \to \mathrm{THM}_T(\ulcorner \underline{0=1} \urcorner)$

From these, (b) follows

qed

Second Incompleteness Theorem: Let T be consistent, axiomatizable, $T \supseteq R$ and satisfies HBL 1-3.

Then $T \nvdash \mathrm{CON}_T$.

Proof Recall D s.t. $R \vdash D \leftrightarrow \neg \mathrm{THM}_T(\ulcorner D \urcorner)$

We'll show: $T \vdash \neg D \to \neg \mathrm{CON}_T$, i.e. $T \vdash \mathrm{CON}_T \to D$

This suffices since $T \nvdash D$

(1) $T \vdash \neg D \to \mathrm{THM}_T(\ulcorner D \urcorner)$ by choice of D.

(2) $T \vdash \mathrm{THM}_T(\ulcorner D \urcorner) \to \mathrm{THM}_T(\ulcorner \mathrm{THM}_T(\ulcorner D \urcorner) \urcorner)$ HBL-2

(3) $T \vdash \mathrm{THM}_T(\ulcorner \mathrm{THM}_T(\ulcorner D \urcorner) \urcorner) \to \mathrm{THM}_T(\ulcorner \neg D \urcorner)$

Pf: By choice of D, $T \vdash \mathrm{THM}_T(\ulcorner D \urcorner) \to \neg D$

So by HBL-1, $T \vdash \mathrm{THM}_T(\ulcorner \mathrm{THM}_T(\ulcorner D \urcorner) \to \neg D \urcorner)$

Use HBL-3 $\mathrm{THM}_T(\ulcorner \mathrm{THM}_T(\ulcorner D \urcorner) \urcorner)$

$\to \mathrm{THM}_T(\ulcorner \mathrm{THM}_T(\ulcorner D \urcorner) \to \neg D \urcorner) \to \mathrm{THM}_T(\ulcorner \neg D \urcorner)$

(4) Therefore $T \vdash \neg D \to \mathrm{THM}_T(\ulcorner \neg D \urcorner)$.

Why does (4) follow from HBL-1?

HBL-1 says $T \vdash \neg D$ then $T \vdash THM_T(\ulcorner \neg D \urcorner)$

This does NOT give us $T \vdash \neg D \to THM_T(\ulcorner \neg D \urcorner)$

$(T \vdash A \Rightarrow T \vdash B) \not\Rightarrow T \vdash (A \to B)$

By (1) + (4)

$$T \vdash \neg D \to (THM_T(\ulcorner \neg D \urcorner) \wedge \neg THM_T(\ulcorner \neg D \urcorner))$$

So $T \vdash \neg D \to \neg CON_T$ by the Lemma, part (b).

Thus $T \vdash CON_T \to D$. Since $T \nvdash D$, $T \nvdash CON_T$.

QED 2nd Incompleteness.

CON_T - true & not provable in T.

Remaining Theorem

Every decidable relation is representable in R.

" Computable function is " " ".

3 part argument:

#1 Basics of representability

- Boolean combination
- Composition
- Bounded Quantifiers $\forall x \leq y (- -)$
- Regular Minimization defined representable functions

$\underline{f(x) = (\mu y)}\ \underline{(R(x,y))}$ $\qquad \mu =$ "the least"

Assume R is decidable

(1) If $\forall x, \exists y, R(x,y)$ then f is computable

If not, f is partial computable

$f(x) = (\mu y)(g(x,y) = 0)$ $\qquad g$ - partial computable

1) $f(x) = y$ means $g(x,y) = 0$ & $\forall z < y \ g(x,z)\downarrow \neq 0$

If $\forall x, f(x)\downarrow$ then this called <u>regular minimization</u>

In this case f is computable

#2 Sequence Coding

$\langle n_1, n_2, n_3 \ldots, n_e \rangle$ sequence of integers

Coded by a single integer

$m := \langle n_1 \ldots n_e \rangle$

Gödel β function $\beta(i, m) = n_i$

There are representable/computable functions to manipulate Gödel numbers of sequences

$2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdot 7^{n_4} \ldots$

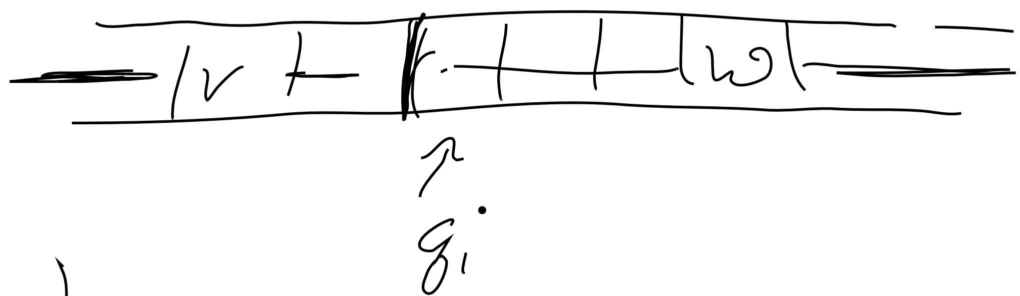
$m = n_1 + n_2 \cdot 2^i + n_3 \cdot 2^{2i} + n_4 \cdot 2^{3i}$

$2^i > n_j$ for all j

$i \mapsto 2^i$ is representable

#3 Coding Turing machine computations with integers

Configuration



$$C = (v, i, w)$$

Complete Computation

halting

$$\langle C_0,\ C_1,\ C_2\ \ldots\ C_e \rangle$$

initial

of configuration

C_{i+1} follows C_i by a single step of the Turing machine