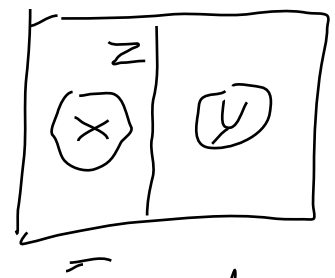
Computably inseparable c.e. sets

<u>Definition</u> Let $X \cap Y = \phi$, $X, Y \subseteq \Sigma^*$ (or $\mathbb{N}$).

X and Y are <u>computably seperable</u> if there is a decidable Z s.t. $X \subseteq Z$ and $Z \cap Y = \phi$ $\quad (Y \subseteq \bar{Z})$

<u>Theorem</u> There are a pair X, Y of computably inseparable c.e. sets

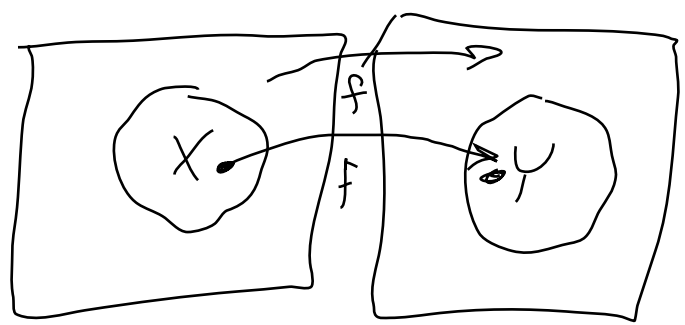


<u>Proof</u>:

$X = \text{Accepts} = \{\ulcorner M \urcorner : M(\epsilon) \text{ accepts}\}$

$Y = \text{Rejects} = \{\ulcorner M \urcorner : M(\epsilon) \text{ rejects}\}$

Many-one reduction



Suppose Z separates X, Y, and Z is decidable.

Let N accept (decide) Z

$N(\ulcorner M \urcorner)$ accepts then $M \notin \text{Rejects}$ so $M(\epsilon)$ does not reject

$N(\ulcorner M \urcorner)$ rejects then $M \notin \text{Accepts}$ so $M(\epsilon)$ does not accept

Let N' decide $\overline{Z}$.

So $N'(\ulcorner M \urcorner)$ accepts $\Rightarrow$ $\ulcorner M \urcorner \notin Accept(\epsilon)$ $\Rightarrow$ $M(\epsilon)$ does not accept.

Similarly $N'(\ulcorner M \urcorner)$ rejects $\Rightarrow$ $M(\epsilon)$ does not reject

Form $D_{N'}$ - by Diagonal Theorem

$D_{N'}(\epsilon)$ accepts $\Leftrightarrow$ $N'(\ulcorner D_{N'} \urcorner)$ accepts

$\Rightarrow$ $D_{N'}$ does not accept

$D_{N'}(\epsilon)$ rejects $\Leftrightarrow$ $N'(\ulcorner D_{N'} \urcorner)$ rejects

$\Rightarrow$ $D_{N'}$ does not reject

<u>Contradiction</u> $N'(\ulcorner D_{N'} \urcorner)$ always accepts or rejects

So $D_{N'}$ always accepts or rejects.

qed

Diagonal Theorem

If M is an algorithm that takes one input, there is an algorithm D_M s.t

$D_M(\epsilon)$ does the same as $M(\ulcorner D_M \urcorner)$

Quine - like property

Incompleteness Theorems – Chapter 9

Basic Theorem Let $\mathcal{N} = (\mathbb{N}, 0, S, +, \cdot)$

$Th\,\mathcal{N}$ is undecidable

Corollary $Th\,\mathcal{N}$ is not $\underbrace{\text{axiomatizable}}$

has a decidable set of axioms

(equivalently, a c.e. set of axioms)

By Theorem: Any axiomatizable, complete theory is decidable

We'll define a very weak theory Q (Robinson's Theory Q)

Theorem: There is no consistent, decidable theory T s.t. $T \supseteq Q$

(Not even required that $T \subseteq Th\,\mathcal{N}$)

Thm Any semidecidable theory is axiomatizable & conversely

We define four theories R, Q, PA, $Th\,\mathfrak{N}$

R, Q - really weak

PA - really strong (axiomatizable) "Peano arithmetic"

All of these theories can "represent" the decidable sets & the computable functions

$\underline{Q}$ language : $0, S, +, \cdot$

Q_1 $\forall x \forall y \, (Sx = Sy \to x = y)$ "injection"

Q_2 $\forall x \, (Sx \neq 0)$ $0 \notin \text{range}(S)$

Q_3 $\forall x \, (x \neq 0 \to \exists y \, (Sy = x))$ range of S is $\mathbb{N} \setminus \{0\}$

Q_4: $\forall x \, (x + 0 = x)$

Q_5: $\forall x \forall y \, (x + Sy = S(x+y))$ } definition of $+$

Q_6: $\forall x \, (x \cdot 0 = 0)$

Q_7 $\forall x \, (x \cdot Sy = x \cdot y + x)$ } definition of $\cdot$

<u>Notation</u> Sx means $S(x)$

$SSS0$,, $S(S(S(0)))$ denotes 3

<u>Fact</u> $Q \nvdash \forall x (0+x = x)$

$Q \nvdash \forall x (Sx \neq x)$

<u>Peano arithmetic</u> (PA)

<u>Definition</u> Let $A = A(x) = A(x, \vec{y})$. The induction axiom for A, denoted Ind_A

$$\forall \vec{y} \left[A(0) \wedge \forall x (A(x) \to A(Sx)) \to \forall x A(x) \right]$$

<u>Example</u> Let A be $x+y = y+x$.

$$\forall y \left[0+y = y+0 \wedge \forall x (x+y = y+x \to Sx+y = y+Sx) \to \forall x (x+y = y+x) \right]$$

Let A be $0+x = x$

$$\underline{0+0=0} \wedge \forall x (\underline{0+x=x \to 0+Sx=Sx}) \to \underline{\forall x (0+x=x)}$$

"Clearly" $PA \subseteq Th\,\mathcal{N}$

But $Q \nvdash 0+x=x$

<u>Example</u> $PA \vdash \forall x\,(0+x=x)$

<u>Def'n</u> PA is the theory axiomatized by $Q_1 - Q_7$ plus Ind_A for all formulas A. with language $L_{PA} = \{0, S, +, \cdot\}$

<u>Pf</u> (a) $Q \vdash 0+0=0$ by Q_4

So $PA \vdash 0+0=0$

(b) $Q \vdash 0+x=x \rightarrow 0+Sx=Sx$

<u>Pf</u> $0+Sx = S(0+x)$ by Q_5

$= Sx$ if $0+x=x$ ✓

(c) By Ind_A when A is $0+x=x$.

$PA \vdash \forall x\,(0+x=x)$

<u>Theory R</u> – Language $0, S, +, \cdot$

$s \leq t$ abbreviation for $\exists x(x+s=t)$

(Not $\exists x(s+x=t)$!)

<u>Infinitely many axioms</u>

Notation $\underline{n}$ means the term $\underbrace{S(S(\cdots S(S(0))\cdots))}_{n \text{ many } S's}$

$\underline{3}$ is $SSS0$

$\underline{4}$ is $SSSS0$

$\underline{0}$ is 0

$R_{\neq}$: $\underline{n} \neq \underline{m}$ for all $n \neq m \in \mathbb{N}$

R_{+}: $\underline{n} + \underline{m} = \underline{n+m}$ for all n, m

$R_{\cdot}$: $\underline{n} \cdot \underline{m} = \underline{n \cdot m}$ " " " "

$R_{\leq 1}$: $\forall x(x \leq \underline{n} \vee \underline{n} \leq x)$ for all $n \in \mathbb{N}$

$R_{\leq 2}$: $\forall x(x \leq \underline{n} \rightarrow x = 0 \vee x = \underline{1} \vee x = \underline{2} \vee \cdots x = \underline{n})$

for all $n \in \mathbb{N}$.

Example

$\underline{2} \neq \underline{3}$ $\qquad$ $SS0 \neq SSS0$

$\underline{2} + \underline{3} = \underline{5}$ $\qquad$ $SS0 + SSS0 = SSSSS0$

$\underline{2} \cdot \underline{3} = \underline{6}$ $\qquad$ $SS0 \cdot SSS0 = SSSSSS0$

$\forall x (x \leq \underline{2} \vee \underline{2} \leq x)$ $\qquad$ $\forall x (x \leq SS0 \vee SS0 \leq x)$

$\forall x (x \leq SS0 \rightarrow x = 0 \vee x = S0 \vee x = SS0)$

$R \vdash \underline{2} + \underline{3} = \underline{5}$ $\qquad$ $\underline{3} + \underline{2} = \underline{5}$ $\qquad$ R_+

$\underline{2} + \underline{3} = \underline{3} + \underline{2}$

$R, Q \vdash \forall x \forall y (x + y = y + x)$

$PA \vdash \forall x \forall y (x + y = y + x)$

Since $Q \vdash \forall x (Sx \neq x)$

there is a nonstandard model $\mathfrak{A}$ of Q

s.t. for some member a of $|\mathfrak{A}|$

$\mathfrak{A} \models Sa = a$