**Computably inseparable c.e. sets**

**Definition** Let \( X \cap Y = \emptyset \), \( X, Y \subseteq \Sigma^* \) (or \( \Sigma \)).

\( X \) and \( Y \) are **computably separable** if there is a decidable \( Z \) s.t. \( X \leq Z \) and \( Z \cap Y = \emptyset \) \( (Y \leq \overline{Z}) \)

**Theorem** There are a pair \( X, Y \) of computably inseparable c.e. sets.

**Proof:**

\[ X = \text{Accepts} = \{ \overline{\langle M, x \rangle} : M(x) \text{ accepts} \} \]

\[ Y = \text{Rejects} = \{ \overline{\langle M, x \rangle} : M(x) \text{ rejects} \} \]

Suppose \( Z \) separates \( X, Y \), and \( Z \) is decidable.

Let \( N \) accept (decide) \( Z \)

\( N(\overline{\langle M, x \rangle}) \) accepts then \( M \not\in \text{Rejects} \) so \( M(f) \) does not reject.

\( N(\overline{\langle M, x \rangle}) \) rejects then \( M \not\in \text{Accepts} \) so \( M(f) \) does not accept.
Let \( N' \) decide \( \overline{Z} \).

So \( N'(\overline{M^2}) \) accepts \( \iff M^2(\text{Accept}) = 1 \) does not accept.

Similarly \( N'(\overline{M^2}) \) rejects \( \iff M(\epsilon) \) does not reject.

Form \( DN' \) by Diagonal Theorem

\[
\begin{align*}
DN'(\epsilon) \text{ accepts } & \iff N'(\overline{DN'}) \text{ accepts} \\
& \Rightarrow DN' \text{ does not accept}
\end{align*}
\]

\[
\begin{align*}
DN'(\epsilon) \text{ rejects } & \iff N'(\overline{DN'}) \text{ rejects} \\
& \Rightarrow DN' \text{ does not reject}
\end{align*}
\]

Contradiction: \( N'(\overline{DN'}) \) always accepts or rejects.

So \( DN' \) always accepts or rejects.

\( \text{qed} \)
Diagonal Theorem

If $M$ is an algorithm that takes one input, there is an algorithm $D_M$ s.t.

$D_M(e)$ does the same as $M(T_{D_M})$

Quine-like property
Incompleteness Theorems - Chapter 9

Basic Theorem  Let $\mathcal{N} = (\mathbb{N}, 0, S, +, 0)$

$\text{Th } \mathcal{N}$ is undecidable

Corollary $\text{Th } \mathcal{N}$ is not axiomatizable

By Theorem: Any axiomatizable, complete theory is decidable

We'll define a very weak theory $Q$ (Robinson's Theory Q)

Theorem: There is no consistent, decidable theory $T$ such that $T \vdash Q$

(Not even required that $T \subseteq \text{Th } \mathcal{N}$.)

Thus, any semidecidable theory is axiomatizable & conversely.
We define four theories \( \mathbb{R}, \mathbb{Q}, \text{PA, Th} \mathbb{N} \).

\( \mathbb{R}, \mathbb{Q} \) - really weak

\( \text{PA} \) - really strong (axiomatizable) "Recursive arithmetic"

All of these theories can "represent" the decidable sets & the computable functions
Language: $0, 1, +, \cdot$

$Q_1: \forall x \forall y (Sx = Sy \to x = y)$

$Q_2: \forall x (Sx \neq 0)$

$Q_3: \forall x (x \neq 0 \to \exists y (Sy = x))$

$Q_4: \forall x (x + 0 = x)$

$Q_5: \forall x \forall y (x + Sy = S(x + y))$

$Q_6: \forall x (x - 0 = 0)$

$Q_7: \forall x (x - Sy = x \cdot y + x)$

\text{"injection"}

0 \notin \text{range}(S)

\text{range of } S \text{ is } \mathbb{N} \setminus \{0\}

\text{definition of } +

\text{definition of } \cdot

\underline{Notation:} \ Sx \text{ means } S(x)

5 \cdot 5 \cdot 5 \cdot 0 = 5(S(5(5(0))))

\text{denotes } 3
Fact  
\[ Q \rightarrow \forall x (0 + x = x) \]
\[ Q \land \forall x (Sx \neq x) \]

Peano arithmetic (PA)

Definition: Let \( A = A(x) = A(x, \bar{y}) \). The induction axiom for \( A \), denoted \( \text{Ind}_A \)

\[ \forall y \left[ A(0) \land \forall x (A(x) \rightarrow A(Sx)) \rightarrow \forall x A(x) \right] \]

Example: Let \( A \) be \( x + y = y + x \).

\[ \forall y \left[ 0 + y = y + 0 \land \forall x \left( x + y = y + x \rightarrow Sx + y = y + Sx \right) \rightarrow \forall x (x + y = y + x) \right] \]

Let \( A \) be \( 0 + x = x \)
\[ 0 + 0 = 0 \land \forall x (0 + x = x \rightarrow 0 + Sx = Sx) \rightarrow \forall x (0 + x = x) \]
"Clearly" \( PA = \mathbb{N} \)

Example \( PA \vdash \forall x \ (0 + x = x) \)

Defn \( PA \) is the theory axiomatized by \( Q_1 - Q_7 \) plus \( Ind_A \) for all formulas \( A \) with language \( \mathcal{L}_{PA} = \{ 0, S, +, \cdot \} \)

Proof \( 2 \) \( Q \vdash 0 + 0 = 0 \) by \( Q_4 \)

\( S_0 \) \( PA \vdash 0 + 0 = 0 \)

\( 3 \) \( Q \vdash 0 + x = x \rightarrow 0 + Sx = Sx \)

Proof \( Q \vdash 0 + Sx = S(0 + x) \) by \( Q_5 \)

\( = Sx \) \( \vdash 0 + x = x \checkmark \)

\( 4 \) By \( Ind_A \) when \( A \vdash 0 + x = x \)

\( PA \vdash \forall x \ (0 + x = x) \)
Theory \( R \) - Language \( 0, S, +, \cdot \)

\( S \leq t \) abbreviation for \( \exists x (x + S = t) \)

(Not \( \exists x (S \cdot x = t) \) !)

Infinitely many axioms

Notation \( n \) means the term \( S(S(\cdots S(S(0))\cdots)) \)

where \( n \) many \( S \)

\( R_\neq : \quad n \neq m \quad \text{for all } n, m \in \mathbb{N} \)

\( R_+ : \quad n + m = n + m \quad \text{for all } n, m \)

\( R_- : \quad n \cdot m = n \cdot m \) (omitted)

\( R_{\leq 1} : \quad \forall x (x \leq n \lor n \leq x) \quad \text{for all } n \in \mathbb{N} \)

\( R_{\leq 2} : \quad \forall x (x \leq n \Rightarrow x = 0 \lor x = 1 \lor x = 2 \lor \ldots \quad x = n) \quad \text{for all } n \in \mathbb{N} \).
Example

\[ 2 + 3 = 5 \]
\[ 5 + 3 = 8 \]
\[ 3 \cdot 3 = 9 \]
\[ 2 \cdot 3 = 6 \]
\[ SS0 \neq SS50 \]
\[ SS0 + SS50 = SS5550 \]
\[ SS50 \cdot SS50 = SS55550 \]

\[ \forall x \ (x \leq 2 \lor 2 \leq x) \]
\[ \forall x \ (x \leq SS0 \lor SS0 \leq x) \]

\[ \forall x \ (x \leq SS0 \rightarrow x = 0 \lor x = SS0 \lor x = SS50) \]

\[ R \vdash \begin{align*}
2 + 3 &= 5 \\
5 + 2 &= 7
\end{align*} \]
\[ R_+ \]
\[ 2 + 3 = 3 + 2 \]

\[ R, Q \vdash \forall x \forall y \ (x + y = y + x) \]
\[ PA + \forall x \forall y \ (x + y = y + x) \]
Since \( Q \vdash A \neg x \ (5x \neq x) \)

there is a non-standard model of \( Q \)

\( A \)

s.t. for some member \( a \) of \( \mathbb{N} \)

\( A \vdash \exists a = a \)