Math 160B - Winter 2022 - January 11.

Cardinalities of structures.

Recall Proof of Completeness Theorem.

<u>Thm</u> (Let L be a countable language.)

If T is consistent, then T is satisfiable.

<u>Proof outline</u>: Start with consistent T.

(1) Add countably many new constant symbols $d_1, d_2, d_3 \ldots$

Extended T to a strongly Henkin, consistent Δ

<u>i.e</u> For $\forall x A(x)$ a sentence

$\rightarrow \begin{cases} \Delta \models A(d_i) \rightarrow \forall x A(x) & \text{for some } d_i \\ \Delta \models \exists x \neg A(x) \rightarrow \neg A(d_i) & d_i \text{ - a} \end{cases}$

~~counterexample~~ witness to $\exists x \neg A(x)$ & a counterexample to $\forall x A(x)$

(1½) Extended Δ to a complete theory Π.

(2) Used equivalence classes of closed terms in $L \cup \{d_1, d_2, \ldots\}$ as the universe of a structure $\mathfrak{A}$.

(3) Then showed $\mathfrak{A} \models \Delta$. ($\mathfrak{A} \models T$)

Since $L \cup \{d_1, d_2, \ldots\}$ is countable, there are countably many terms, so $|\mathfrak{A}|$ is countable.

In Δ: $A(d_i) \to \forall x A(x)$

$\exists x \neg A(x) \to \neg A(d_i)$

$\Delta \vdash$ "If $\forall x A(x)$ fails, then $A(d_i)$ fails"

d_i - a counterexample to $\forall x A(x)$.

In Δ: $\exists x A(x) \to A(d_j)$ for some j

d_j is a "witness" for $\exists x A(x)$.

<u>Compare to generalization</u> (See also Theorem on Constants)

Γ - set of sentences.

If $\Gamma \models A(x)$

then $\Gamma \models \forall x A(x)$

Theorem on constants If $\Gamma \models A(d)$ for d a new constant

then $\Gamma \models \forall x A(x)$

Strongly Henkin is based on:

$\Gamma \cup \{A(d) \to \forall x(x)\}$ is consistent

if d is "new" & Γ consistent

Completeness theorem for a Countable Language L

If Γ is a consistent set of L-sentences, then Γ has a countable model.

Cardinalities

(1) Finite
(2) Countably infinite
(3) Uncountable infinite

} Countable

$0 \quad 1 \quad 2 \quad 3 \quad \cdots$

$\aleph_0 \quad \aleph_1 \quad \aleph_2 \quad \cdots \quad c$

$\omega \qquad\qquad \tau$

c = continuum cardinality of $\mathbb{R}$

aleph

Corollary Th R has a countable model.

$$R = (\mathbb{R}, 0, 1, +, \cdot, <)$$

Corollary Let ZF be Zermelo-Fraenkel set theory $\{\in\}$

(If ZF is consistent) ZF has a countable model.

"Skolem paradox"

$Th\,\mathbb{R}$ = theory of real closed fields RCF

a "smallest" model of RCF is the real closure of $\mathbb{Q}$.

Completeness Theorem for (Uncountable) languages L

<u>Def'n</u> $\mathrm{card}(L) = \max\{|L|, \aleph_0\} = \begin{cases} |L| & \text{if } L \text{ is infinite} \\ \aleph_0 & \text{if } L \text{ is finite.} \end{cases}$

So $\mathrm{card}(L)$ is equal to the number of L-formulas.

<u>Notation</u> $|X|$ - cardinality of X.

<u>Theorem</u> If Γ is a consistent set of L-sentences, then Γ has a model of cardinality $\leq \mathrm{card}(L)$.

<u>Pf</u> - See exercise in the text.

Theorem:

① If T has arbitrarily large finite models, then T has an infinite model.

② If T has an infinite model, then T has a model of cardinality λ for every $\lambda \geq \mathrm{card}(L)$. (Löwenheim-Skolem theorem)

Corollary to this theorem

Th $\mathcal{N}$ has an uncountable (hence, nonstandard) model.

Corollary to earlier theorems (and their proof)

Th $\mathcal{N}$ has a countable nonstandard model.

Proof ideas ① - already did - same proof works.

② Add new constants d_γ for $\gamma < \lambda$ (ie λ many values γ)

Add to T the sentences $d_\gamma \neq d_{\gamma'}$ for $\gamma \neq \gamma'$.

The result is finitely satisfiable, hence satisfiable.

Any model of $T \cup \{d_\gamma \neq d_{\gamma'} : \gamma \neq \gamma'\}$ must have cardinality $\geq \lambda$.

and by Completeness Theorem it has a model of cardinality $\leq \lambda$.

Hence it a model of exactly λ. qed

Countable Łos-Vaught:

Theorem: Let T be a theory in a countable language. Suppose T has no finite models.
If T is $\aleph_0$-categorical, the T is complete.

Defn T is κ-categorical if all models of T of cardinality κ are isomorphic.

So T is $\aleph_0$-categorical iff T has only one countably infinite model up to isomorphism.

Pf Suppose T is not complete. So $T \cup \{A\}$ and $T \cup \{\neg A\}$ are both consistent for some sentence A.
Let $\mathfrak{A}$, $\mathfrak{B}$ be countable models of $T \cup \{A\}$ and $T \cup \{\neg A\}$.
$\mathfrak{A} \models T$ and $\mathfrak{B} \models T$, so $\mathfrak{A} \cong \mathfrak{B}$ since T is $\aleph_0$ categorical and since T has no finite models.
Contradiction. #

Next Finish discussing Chapter V on Thursday
Start algorithms in Chapter 7.

General Łos-Vaught ~~Theo~~ Test $\quad t \quad \setminus L$

Theorem If T has no ~~countable~~ finite models and T is λ-categorical for some $\lambda \geq \text{card}(L)$ then T is complete.

Proof Almost identical to previous proof.

Let T be $\{$"at least k objects such that $P(x)$"$\}$
$\cup \{$"at least k objects such that $\neg P(x)$"$\}$

For K an uncountable cardinal, there are non isomorphic models

T is $\aleph_0$-categorical – and hence T is complete

Halmos - Naive Set Theory.

Kenneth Kunen - UG text + a graduate text on set theory.