Cardinalities of structures.

Recall Proof of Completeness Theorem.

Then (Let $L$ be a countable language.)

If $T$ is consistent, then $T$ is satisfiable.

Proof outline: Start with consistent $T$.

1. Add countably many new constant symbols $d_1, d_2, d_3, \ldots$

   Extended $T$ to a strongly Henkin, consistent $\Delta$

   i.e. for $\forall x \ A(x)$ a sentence

   $\rightarrow \bigwedge \Delta \models A(d_i) \rightarrow \forall x \ A(x)$ for some $d_i$.

2. Extended $\Delta$ to a complete theory $\bar{T}$.

3. Used equivalence classes of closed terms

   in $L \cup \{d_1, d_2, \ldots\}$ as the universe of a structure $\mathcal{A}$.

4. Then showed $\mathcal{A} \models \Delta$, ($\mathcal{A} \models T$).

Since $L \cup \{d_1, d_2, \ldots\}$ is countable, there are countably many terms, so $|\mathcal{A}|$ is countable.
In $\mathcal{D}$:

$A(d_i) \rightarrow \forall x A(x)$

$\exists x \neg A(x) \rightarrow \neg A(d_i)$

$\Delta \equiv \text{"If } \forall x A(x) \text{ fails, then } A(d_i) \text{ fails"}$

$d_i$ - a counterexample to $\forall x A(x)$.

In $\Delta$:

$\exists x A(x) \rightarrow A(d_j)$ for some $j$

$d_j$ is a "witness" for $\exists x A(x)$.

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**Comprehension by Generalization** (See also Theorem on Constants)

If $T \models A(x)$

then $T \models \forall x A(x)$

Theorem on constants:

If $T \models A(d)$ for a new constant $d$

then $T \models \forall x A(x)$

Strongly Henkin is based on:

$T \cup \exists A(d) \rightarrow \forall x A(x)$ is consistent

if $d$ is "new" in $T$.
Completeness theorem for a countable language $L$

If $\Gamma$ is a consistent set of $L$-sentences, then $\Gamma$ has a countable model.

Cardinalities

1. Finite
2. Countably infinite
3. Uncountable infinite

\[0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots, \aleph_\omega, \aleph_\kappa, \ldots, \aleph_\delta, \ldots, \aleph_\omega, \ldots, \]

$\aleph_\kappa$ = continuum

Corollary

The $\mathbb{R}^n$ has a countable model.

$\mathbb{R} = (\mathbb{R}, 0, 1, +, 0, <)$

Corollary

Let ZF be Zermelo-Frankel set theory $\{\mathcal{E}\}$

(If ZF is consistent) ZF has a countable model.

"Skolem paradox"
The theory of real closed fields RCF.

A "smallest" model of RCF is the real closure of \( \mathbb{Q} \).

Completeness Theorem for (Uncountable) languages \( L \)

Define \( \text{card}(L) = \max\{ |L|, \mathcal{N}_0 \} = \{ |L| \text{ if } L \text{ is infinite} \}
\mathcal{N}_0 \text{ if } L \text{ is finite}. \)

So \( \text{card}(L) \) is equal to the number of \( L \)-formulas.

Notation \( |X| \) - cardinality of \( X \).

Theorem If \( T \) is a consistent set of \( L \)-sentences,
then \( T \) has a model of cardinality \( \leq \text{card}(L) \).

Pf - See exercise in the text.
Theorem:

1. If $T$ has arbitrarily large finite models, then $T$ has an infinite model.

2. If $T$ has an infinite model, then $T$ has a model of cardinality $\lambda$ for every $\lambda > \text{card}(L)$. (Löwenheim-Skolem theorem)

Corollary to this theorem

$\mathbb{N}$ has an uncountable (hence, nonstandard) model.

Corollary to earlier theorems (and their proof)

$\mathbb{N}$ has a countable nonstandard model.

Proof ideas 1) already did same proof work.

2) Add new constants $d_\gamma$ for $\gamma < \lambda$ (ie. $\lambda$ many values $\gamma$)

Add to $T$ the sentences $d_\gamma \neq d_\gamma$ for $\gamma \neq \gamma'$.

The result is finitely satisfiable, hence satisfiable.

Any model of $T \cup \{d_\gamma \neq d_\gamma : \gamma \neq \gamma'\}$ must have cardinality $\geq \lambda$. 

and by Completeness Theorem it has a model of cardinality $\leq \aleph_0$.

Hence it a model of exactly $\forall$. 

\textbf{Countable Los-Vaught:}

\underline{Theorem:} Let $T$ be a theory in a countable language.
Suppose $T$ has no finite models.
If $T$ is $\forall_0$-categorical, then $T$ is complete.

\underline{Defn:} $T$ is $K$-categorical if all models of $T$ of cardinality $K$ are isomorphic.

So $T$ is $\forall_0$-categorical if $T$ has only one countably infinite model up to isomorphism.

\underline{Pf:} Suppose $T$ is not complete. So $T \cup \{A\}$ and $T \cup \{\neg A\}$ are both consistent for some sentence $A$.
Let $M$, $N$ be countable models of $T \cup \{A\}$ and $T \cup \{\neg A\}$.
$M \models T \land \neg A$, $N \models T \land A$, so $M \equiv N$ since $T$ is $\forall_0$-categorical and since $T$ has no finite models.
Next Finish discussing Chapter V on Thursday. Start algorithms in Chapter 7.

General Tarski-Vaught Test

Theorem: If $T$ has an infinite model and $T$ is $\lambda$-categorical for some $\lambda \geq \text{card}(L)$, then $T$ is complete.

Proof: Almost identical to previous proof.

Let $T$ be $\{ \text{"at least } k \text{ objects such that } P(x) \}$

$U \cup \{ \text{"at least } k \text{ objects such that } \neg P(x) \}$

For $k$ an uncountable cardinal, there are non-isomorphic models $T$ is $\lambda$-categorical — and hence $T$ is complete.
Halmos - Naive Set Theory.

Kenneth Kunen - UG text + a
Graduate text on set theory.