Definition: A theory is a set of sentences closed under logical consequences (for sentences).

If $T$ is a theory and $T \models A$, then $A \in T$.

Example: Theory of groups.

(Finitely axiomatized, elementary class, EC)

Theorem: Let $T$ be a theory and suppose $T$ has arbitrarily large finite models. (For all $k \in \mathbb{N}$, $T$ has a model of cardinality $\geq k$.)

Then $T$ has an infinite model.

"Over spill property" (spill over from arbitrarily large finite to infinite)

Proof: $\exists \overline{x}$ \text{ At least } k sentence says "there are at least $k$ objects"

Use the fact that $T \cup \{ \text{At least } k \geq 2 \}$ is (finitely) satisfiable. It's finitely satisfiable since $T$ has arbitrarily large models.

By Compactness, it is satisfiable. ("It" $= T \cup \{ \text{At least } k \geq 2 \}$)

Any model of $T \cup \{ \text{At least } k \geq 2 \}$ is an infinite model of $T$. \[\Box\]
Nonstandard model of the integers (i.e. the nonnegative integers)
\[ \mathbb{N} = (\mathbb{N}, 0, S, +, -) \quad \mathbb{N} = \{0, 1, 2, \ldots \} \]

**Definition**
Th \( \mathbb{N} \) - \( \mathcal{L} \)-sentence \( A : \mathbb{N} \models A \). Set of sentences true in \( \mathbb{N} \).

\( \mathbb{N} \)-standard model of the integers.

A nonstandard model of the integers - a model \( \mathcal{M} \) of Th(\( \mathbb{N} \)) such that \( \mathcal{M} \) is not isomorphic to \( \mathbb{N} \).

**Theorem**
There is a nonstandard model of the integers.

**Proof**: \( \mathcal{M} \) is isomorphic to \( \mathbb{N} \), written \( \mathcal{M} \cong \mathbb{N} \), if they are the same “up to identity of objects in the universe”.

**Formally**
There is a bijection \( \pi : \{a\} \rightarrow \mathbb{N} \)

1. For every \( c \in \mathcal{L} \), \( \pi(c) \) is a \( \mathbb{N} \).
2. For every \( f \in \mathcal{L} \), and every \( a_1, \ldots, a_k \in \mathbb{N} \),
   \[ \pi(f^\mathcal{M}(a_1, \ldots, a_k)) = f^\mathbb{N}(\pi(a_1), \ldots, \pi(a_k)) \]
3. For every \( R \in \mathcal{L} \), \( a_1, \ldots, a_k \in \mathbb{N} \),
   \[ R^\mathcal{M}(a_1, \ldots, a_k) \iff R^\mathbb{N}(\pi(a_1), \ldots, \pi(a_k)) \]
Theorem. If $\mathcal{M} \sqsubseteq \mathcal{F}$ and $\mathcal{C}$ is an $\mathcal{L}$-sentence then $\mathcal{M} \models C$ iff $\mathcal{F} \models C$.

Definition. $\mathcal{M}$ is elementarily equivalent to $\mathcal{F}$, written $\mathcal{M} \equiv \mathcal{F}$, iff $\forall C, \mathcal{M} \models C \iff \mathcal{F} \models C$, i.e. $\text{Th } \mathcal{M} = \text{Th } \mathcal{F}$.

Restated Theorem. If $\mathcal{M} \equiv \mathcal{F}$, then $\mathcal{M} \equiv \mathcal{F}$.

Proof of existence of nonstandard model.

Let $\mathcal{K}$ be a new constant symbol.

Let $T = \text{Th } \mathcal{K} \cup \{ \mathcal{C} \neq 0, \mathcal{C} 
eq 1, \mathcal{C} \neq 2, \mathcal{C} \neq 3 \}$

$1 \equiv S(0)$,
$2 \equiv S(S(0))$,
$3 \equiv S(S(S(0)))$,

- closed term that represent integers ("numerals")

Then $T$ is finitely satisfiable.

Let any finite subset have only finitely many $\mathcal{C} \neq k$'s.

Use the standard model $\mathcal{N}$, let $\mathcal{C}^a$ be some other value than one of these $k$'s.
Continuing proof

$T$ is satisfiable by Compactness Theorem.

So $T$ has a model $\mathfrak{a}$. 

Claim $\mathfrak{a} \not\models \Pi$. 

Any $\pi: \mathbb{N} \to \mathfrak{a}$ must have $\pi(1) = S^\mathfrak{a}(S^\mathfrak{a}(\ldots))$. 

So $c^\mathfrak{a}$ is not in the range of $\pi$. 

Contradiction.

Q.E.D.

Since $\mathbb{N} \models \forall x \exists y (y + y = x \lor S(y + y) = x)$

in $\mathfrak{a}$, there is $b \in \mathfrak{a}$, $b + b = c$ or $S(b + b) = c$

$c$ is sometimes called a hyper-finite integer. 

Later Incompleteness Theorem - impossibility of giving a complete set of axioms of $\mathfrak{a}$. 

(For a non-standard integer)
In a, \( S^n(c^a) \in 1a \)

\[ d = S^n(c^a), \quad d = 0, \quad d = 1, \quad d \neq 2 \]

\[ \mathbb{N} = \forall x \left( x > k \Rightarrow S(x) > k+1 \right) \]

So, \( S^a/c^a \) is also non-standard.

**Definition:**

Next time, \( A \) can be chosen to be countable.
Def: Let \( L \leq L' \) be languages.

Let \( \mathcal{A} \) be an \( L \)-structure

Let \( \mathcal{B} \) be an \( L' \)-structure

\( \mathcal{A} \) is the \underline{restriction} of \( \mathcal{B} \) to \( L \) if

\[ \sigma \subseteq |\mathcal{A}| \]

\[ |\mathcal{A}| = |\mathcal{B}| \]

\( f^\mathcal{B}, c^\mathcal{B}, R^\mathcal{B} \) are the same as

\( f^\mathcal{A}, c^\mathcal{A}, R^\mathcal{A} \) for all \( f, c, R \in L \).

In this case, we say \( \mathcal{B} \) is an expansion of \( \mathcal{A} \).

\[ \text{Next time: Cardinalities.} \]

\[ \text{Review proof of Completeness Theorem.} \]

\[ \text{Look at section IV.5} \]