Def'n A <u>theory</u> is a set of sentences closed under logical consequences. (for sentences)

If T is a theory and $T \models A$, then $A \in T$.

Example Theory of groups.

(Finitely axiomatized, elementary class, EC)

Theorem: Let T be a theory & suppose T has arbitrarily large finite models. (For all $k \in \mathbb{N}$, T has a model of cardinality $\geq k$.)

Then T has an infinite model.

"Overspill property" (Spill over from arbitrarily large finite to infinite)

Proof: AtLeast_k - sentence says "there are at least k objects"

Use the fact that $T \cup \{\text{AtLeast}_k : k \geq 2\}$ is (finitely) satisfiable.

Its finitely satisfiable since T has arbitrarily large models

By Compactness, it is satisfiable. ("It" $= T \cup \{\text{AtLeast}_k\}$.)

Any model of $T \cup \{\text{AtLeast}_k\}_k$ is an infinite model of T. $\boxed{\checkmark}$

Nonstandard model of the integers (i.e. the nonnegative integers)

$\mathcal{N} = (\mathbb{N}, 0, S, +, \cdot)$ $\quad \mathbb{N} = \{0, 1, 2, \dots\}$ $\quad L = \{0, S, +, \cdot\}$

Defn $Th\,\mathcal{N} = \{L\text{-sentence } A : \mathcal{N} \models A\}$. Set of sentences true in $\mathbb{N}$.

$\mathcal{N}$ - standard model of the integers.

A nonstandard model of the integers - a model $\mathfrak{A}$ of $Th(\mathcal{N})$ such that $\mathfrak{A}$ is not isomorphic to $\mathcal{N}$.

Theorem There is a nonstandard model of the integers.

~~Proof~~: $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$, written $\mathfrak{A} \cong \mathfrak{B}$, if they are the same "up to identity of objects in the universe".

Formally There is a bijection $\pi : |\mathfrak{A}| \to |\mathfrak{B}|$

① For every $c \in L$, $\pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$

② For every $f \in L$, and every $a_1 \dots a_k \in |\mathfrak{A}|$,

$$\pi(f^{\mathfrak{A}}(a_1 \dots a_k)) = f^{\mathfrak{B}}(\pi(a_1), \dots, \pi(a_k))$$

③ For every $R \in L$, $a_1 \dots a_k \in |\mathfrak{A}|$

$$R^{\mathfrak{A}}(a_1 \dots a_k) \Leftrightarrow R^{\mathfrak{B}}(\pi(a_1), \dots, \pi(a_k))$$

Theorem If $\mathfrak{A} \cong \mathfrak{B}$ and C is an L-sentence then $\mathfrak{A} \models C$ iff $\mathfrak{B} \models C$.

Definition $\mathfrak{A}$ is <u>elementarily equivalent</u> to $\mathfrak{B}$, written

$\mathfrak{A} \equiv \mathfrak{B}$, iff $\forall C$, $\mathfrak{A} \models C \Leftrightarrow \mathfrak{B} \models C$, (equiv)

i.e. $\mathrm{Th}\,\mathfrak{A} = \mathrm{Th}\,\mathfrak{B}$

Restate Theorem If $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$

Proof of existence of nonstandard model:

$\mathrm{Th}\,\mathfrak{N}$ Let c be a new constant symbol.

Let $T = \underline{\mathrm{Th}\,\mathfrak{N}} \cup \{c \neq 0, c \neq \underline{1}, c \neq \underline{2}, c \neq \underline{3}, \ldots\}$

$\underline{1}$ is $S(0)$, $\underline{2}$ is $S(S(0))$, $\underline{3}$ is $S(S(S(0)))$, ..

- closed term that represent integers ("numerals")

Claim T is finitely satisfiable.

Pf Any finite subset has only finitely many $c \neq \underline{k}$'s

Use the standard model $\mathfrak{N}$, let $c^{\mathfrak{A}}$ be some other value than one of these k's

Continuing proof

T is satisfable by Compactness Theorem

So T has a model $\mathfrak{A}$.

Claim $\mathfrak{A} \not\cong \mathfrak{N}$.

Any $\pi : \mathfrak{N} \to \mathfrak{A}$ must have $\pi(i) = \underbrace{S^{\mathfrak{A}}(S^{\mathfrak{A}} \cdots S^{\mathfrak{A}}(0)\cdots)}_{i}$

So $c^{\mathfrak{A}}$ is not in the range of π.

Contradiction.

Q.E.D.

Since $\mathfrak{N} \models \forall x \exists y (y+y=x \vee S(y+y)=x)$

in $\mathfrak{A}$, there is $b \in |\mathfrak{A}|$, $b+b=c$ or $S(b+b)=c$

c is sometimes called a hyper finite integer. (or a nonstandard integer)

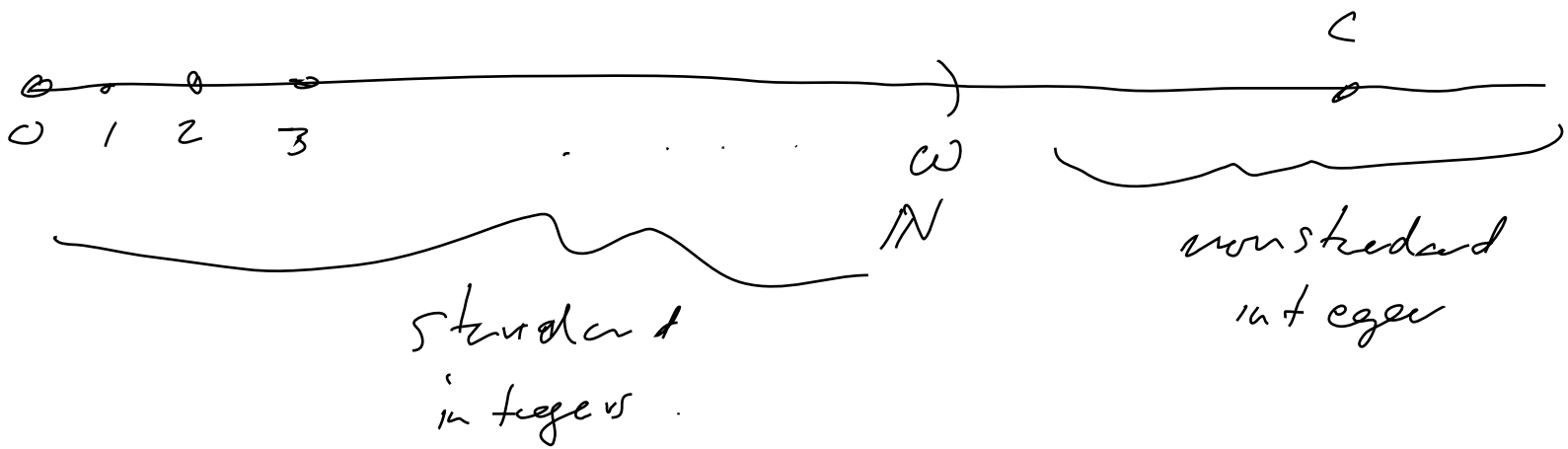
Later Incompleteness Theorem - impossibility of giving a complete set of axioms of Th $\mathfrak{N}$.

In $\mathfrak{A}$, $\quad S^{n}(c^{\mathfrak{A}}) \in |\mathfrak{A}|$

$d = S^{n}(c^{\mathfrak{A}})$. $\qquad d=0,\ d=\underline{1},\ d\neq\underline{2}\ \ldots$

$\mathfrak{N} \models \forall x\,(x > \underline{k} \to S(x) > \underline{k+1})$

So $S^{n}(c^{\mathfrak{A}})$ is also nonstandard.



~~Definition~~:

Next time $\mathfrak{A}$ can be chosen to be countable.

Defn Let $L \subseteq L'$ be languages.

Let $\mathfrak{A}$ be an L-structure

Let $\mathfrak{B}$ be an L'-structure

$\mathfrak{A}$ is the restriction of $\mathfrak{B}$ to L if

restriction

(a) $|\mathfrak{A}| = |\mathfrak{B}|$

(b) $f^{\mathfrak{A}}$, $c^{\mathfrak{A}}$, $R^{\mathfrak{A}}$ are the same as

$f^{\mathfrak{B}}$, $c^{\mathfrak{B}}$, $R^{\mathfrak{B}}$ for all $f, c, R \in L$.

In this case, we say $\mathfrak{B}$ is an <u>expansion</u> of $\mathfrak{A}$.

Next time Cardinalities.

Review proof of Completeness Theorem

Look at section V.5