Topics

- Applications of Compactness & Completeness
  IV.11 (maybe IV.12), V.3, V.4, V.5
  1st topic
  Review of proof of Completeness Theorem.
- Computability - Informal, Turing Machines, Recursive Functions
- Peano arithmetic + other theories of arithmetic.
- Incompleteness Theorems
- Other topics? Herbrand's Theorem, Finite Model Theory,

Section V.11 + especially V.3

Completeness and Soundness Theorems: (T - set of sentences)

T is satisfiable if and only if T is consistent.

Compactness Theorem:

T is satisfiable if and only if every finite subset of T is satisfiable

"T is finitely satisfiable"

Def'n Let $\mathfrak{A}$ be a L-structure.

The size or cardinality of $\mathfrak{A}$ is the cardinality of $|\mathfrak{A}|$.

Theorem A (Theorem V.43). Let L be a language that contains $=$.

There is no sentence B which exactly characterizes the finite L-structures.

I.e there is no B s.t. $\forall$ L-structures $\mathfrak{A}$, ($\mathfrak{A}$ is finite iff $\mathfrak{A} \vDash B$)

Proof of Theorem A:

Recall AtLeast_k is a sentence that's true in $\mathfrak{A}$ iff $|\mathfrak{A}| \geq k$.

Suppose there is such a B

Form $T = \{B\} \cup \{\text{AtLeast}_k : k \geq 2\}$.

Claim T is finitely satisfiable.

Pf Take any finite subset Δ of T, $\Delta \subseteq \{B\} \cup \{\text{AtLeast}_k\}_{k \leq k_0}$

Take an L-structure of size k_0.

This structure satisfies Δ.

Therefore T is satisfiable (by Compactness)

So T has a model $\mathfrak{A}$.

- $\mathfrak{A}$ is infinite since $\mathfrak{A} \models \text{AtLeast}_k, \forall k$
- $\mathfrak{A} \models B$ since $B \in T$.

This is a contradiction.

QED.

Def'n Fix a language L. Let $\mathcal{S}$ be a class of L-structures.

$\mathcal{S}$ is an elementary class if there is a sentence β such that

$\forall$ L-structures $\mathfrak{A}$, $(\mathfrak{A} \in \mathcal{S} \iff \mathfrak{A} \models \beta)$.

Language L contains $=$

Theorem A: The class of finite L-structures is not an elementary class.

Terminology $\mathcal{S}$ is EC means $\mathcal{S}$ is an elementary class

Def'n $\mathcal{S}$ is an elementary class in the wide sense (is EC_Δ)

if there is a set of sentences T such that

$\forall$ L-structures $\mathfrak{A}$ $(\mathfrak{A} \in \mathcal{S} \iff \mathfrak{A} \models T)$.

Notation: Mod T is the class of models of T.

$\mathcal{S}$ is EC_Δ if $\mathcal{S} = \text{Mod } T$ for some set of sentences T.

Example Let T be the ~~three~~ three axioms for groups

Mod T is the class of all groups.

Theorem B: The class of finite $\mathcal{L}$-structures is not EC_Δ

I.e. $\neg\exists$ set Γ of sentences Γ s.t. $\mathrm{Mod}\,\Gamma$ is the re— of finite $\mathcal{L}$-structures.

This will be a corollary of:

Theorem C: If $\mathcal{S}$ and $\mathcal{T}$ are ~~dis~~ complementary classes of $\mathcal{L}$-structures, i.e. $\mathcal{S} \cap \mathcal{T} = \emptyset$ and $\mathcal{S} \cup \mathcal{T}$ is the class of all $\mathcal{L}$-structures

And if $\mathcal{S}$ and $\mathcal{T}$ are EC_Δ then $\mathcal{S}$ and $\mathcal{T}$ are EC.

Take $\mathcal{S}$:= class of finite $\mathcal{L}$-structures
$\mathcal{T}$:= class of infinite $\mathcal{L}$-structures.
$\Big\}$ $\mathcal{S}$, $\mathcal{T}$ are complementary

$\mathcal{S}$ is not EC by Theorem A.

$\mathcal{T}$ is EC_Δ by $\Gamma = \{\mathrm{AtLeast}_k : k \geq 2\}$. $\mathrm{Mod}\,\Gamma = \mathcal{T}$

Proof of Theorem C:

Let $\mathcal{S} = \mathrm{Mod}\ \Gamma$

$\mathcal{T} = \mathrm{Mod}\ \Delta$

Then $\Gamma \cup \Delta$ is unsatisfiable, since $\mathcal{S} \cap \mathcal{T} = \emptyset$

So there finite $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$ s.t. $\Gamma_0 \cup \Delta_0$ is unsatisfiable (by Compactness!)

Let $A = \bigwedge \Delta_0$ $\qquad \bigwedge$ - conjunction $\qquad \Sigma : + :: \bigwedge : \wedge$

$B = \bigwedge \Gamma_0$

Claim: $\mathcal{S} = \mathrm{Mod}\ B \qquad \mathcal{T} = \mathrm{Mod}\ A$

Pf Any structure $\mathfrak{A}$ is a model of either Γ or Δ but not both

hence a model of either Γ_0 or Δ_0 but not both ($\Gamma_0 \cup \Delta_0$ is unsatisfiable)

The structure thus is in $\mathcal{S}$ iff it satisfies Γ_0 and thus B

" " $\mathfrak{A}$ is in $\mathcal{T}$ iff it satisfies Δ_0 and thus A.

QED.

Reading for Thursday

(1) Read more in V.4 (go back to IV.11 as needed)

(2) Review the proof the Completeness in V.3

- be ready to ask questions about it