Problems 1 and 5 depend on the proof of the Gödel Second Incompleteness Theorem which will be covered in class on Tuesday (March 8).

1. Let \( A \) be an arbitrary \( L_{PA} \)-sentence. Prove that \( PA \) does not prove \( \neg \text{Thm}_{PA}(\overline{\text{Thm}}_P A) \).

2. Let \( E \) be a self-referential formula such that \( R \) proves \( E \iff \text{Thm}_{PA}(\overline{\text{Thm}}_P E) \). Is \( E \) true or false? Justify your answer.

3. Let \( F \) be a self-referential formula such that \( R \) proves \( F \iff \text{Thm}_{PA}(\overline{\text{Thm}}_P F) \). Is \( F \) true or false? Justify your answer.

4. Let \( T \supseteq R \) be a consistent, axiomatizable theory. This exercise asks you to carry Rosser’s construction of an independent sentence. A Rosser proof of a formula \( A \) is defined to be a proof \( P \) of \( A \) such that there is no proof \( P' \) of \( \neg A \) with \( \overline{P'} < \overline{P} \). More formally, let \( \text{Neg}(x_1,x_2) \) represent the mapping \( \overline{A} \iff \overline{\neg A} \) in \( R \) and define \( \text{Rprf}_T(w,x) \) to be the formula

\[
\text{Rprf}_T(w,x) := \text{Prf}_T(w,x) \land \forall v \forall y (v < w \land \text{Neg}(x,y) \rightarrow \neg \text{Prf}_T(v,y)).
\]

Let \( \text{Rthm}_T(x) \) be the formula \( \neg \exists w \text{Rprf}_T(w,x) \). Finally, let \( D_R \) be a self-referential formula such that \( R \) proves \( D_R \iff \overline{\text{Rthm}}_T(\overline{D_R}) \). Also, let \( A \) be an arbitrary formula.

(a) There is a Rosser \( T \)-proof of \( A \) if and only if there is a \( T \)-proof of \( A \).

(b) If \( T \vdash A \), then \( T \vdash \text{Rthm}_T(\overline{A}) \).

[Hint: You will need to use \( T \supseteq R \) and Axiom \( R'_1 \).]

(c) If \( T \vdash \neg A \), then \( T \vdash \neg \text{Rthm}_T(\overline{A}) \).

(d) Prove that \( T \nvdash D_R \).

(e) Prove that \( T \nvdash \neg D_R \).

(f) Conclude that \( D_R \) is independent of \( T \).

5. Suppose \( T \supseteq R \) is consistent, axiomatizable and satisfies the Hilbert-Bernays-Löb conditions. Let \( D \) be a self-referential formula such that \( R \vdash D \iff \overline{\text{Thm}}_T(\overline{D}) \).

(a) Prove \( T \vdash D \iff \text{CON}_T \).

(b) Suppose that \( E \) is another sentence such that \( R \vdash E \iff \overline{\text{Thm}}_T(\overline{E}) \). Prove that \( T \vdash D \iff E \).

(c) Show that there is a theory \( T \) such that \( T \vdash \neg D \). (This shows that the assumption of \( \omega \)-consistency in Theorem IX.31 cannot be weakened to the assumption of consistency.)