Math 160B - Winter 2022 - Homework 5
Due Monday, February 7, 11:00am
(Hand in by uploading to Gradescope)

1. Give the state diagram for a Turing machine that uses only the tape symbols $\Gamma = \{0, 1, \#\}$ and computes the symbol-doubling function

$$a_1 a_2 \cdots a_{n-1} a_n \mapsto a_1 a_1 a_2 a_2 \cdots a_{n-1} a_{n-1} a_n a_n$$

where 0's are replaced with "00", and 1's with "11".

2. Prove that there is no Turing machine M such that: (a) If M is started with the input tape is completely blank, then M accepts, and (b) If there is a non-blank symbol anywhere on the tape when M starts, then M rejects.

3. Define Total to be the set

$$\text{Total} = \{\ulcorner M \urcorner : \text{Algorithm } M \text{ halts on all inputs } w\}.$$

In other words, Total is the set of Gödel numbers $\ulcorner M \urcorner$ such that M computes a (total) function.

(a) Give a many-one reduction from Halt_0 to Total.

(b) Give a many-one reduction from $\overline{\text{Halt}_0}$ to Total. $\overline{\text{Halt}_0}$ denotes the complement of Halt_0.

(c) Prove that Total is neither c.e. nor co-c.e.

4. (The Busy Beaver function for running time.) For this problem and the next problem, restrict attention to Turing Machines over the alphabets $\Sigma = \Gamma = \{1, \$\}$. Define the running-time version of the Busy Beaver function $\text{BB}_{\text{steps}} : \mathbb{N} \to \mathbb{N}$ by

$$\text{BB}_{\text{steps}}(n) = \max\{m \geq 0 : \text{there is a Turing Machine } M \text{ with } n \text{ states such that } M(\epsilon) \text{ halts after exactly } m \text{ steps.}\}$$

(a) Prove that $\text{BB}_{\text{steps}}(n)$ is not computable by proving that otherwise $\text{Halt}_0^{\text{TM}}$ would be decidable. $\text{Halt}_0^{\text{TM}}$ is the version of Halt_0 that applies to algorithms which are specified as Turing machines.

(b) Suppose $f : \mathbb{N} \to \mathbb{N}$ is computable. Prove that $\text{BB}_{\text{steps}}(n)$ eventually dominates f, namely there is an N such that $f(n) < \text{BB}_{\text{steps}}(n)$ for all $n > N$.

5. (Busy Beaver function for output value.) Use the same conventions as the previous problem. The Busy Beaver function $\text{BB} : \mathbb{N} \to \mathbb{N}$ is defined by

$$\text{BB}(n) = \max\{m \geq 0 : \text{There is a Turing Machine } M \text{ with } n \text{ states such that } M(\epsilon) \text{ halts with } m \text{ many 1's written on the tape.}\}$$

Suppose $f : \mathbb{N} \to \mathbb{N}$ is computable. Prove that $\mathrm{BB}(n)$ eventually dominates f. Conclude that BB is not computable.