1. Give the state diagram for a Turing machine that uses only the tape symbols $\Gamma = \{0, 1, \#\}$ and computes the symbol-doubling function

$$a_1a_2\ldots a_{n-1}a_n \mapsto a_1a_1a_2a_2\ldots a_{n-1}a_{n-1}a_na_n$$

where 0’s are replaced with “00”, and 1’s with “11”.

2. Prove that there is no Turing machine $M$ such that: (a) If $M$ is started with the input tape is completely blank, then $M$ accepts, and (b) If there is a non-blank symbol anywhere on the tape when $M$ starts, then $M$ rejects.

3. Define Total to be the set

$$\text{Total} = \{ \# M : \text{Algorithm } M \text{ halts on all inputs } w \}.$$  

In other words, Total is the set of Gödel numbers $\# M$ such that $M$ computes a (total) function.

(a) Give a many-one reduction from $\overline{\text{Halt}_0}$ to Total.

(b) Give a many-one reduction from $\text{Halt}_0$ to Total. $\text{Halt}_0$ denotes the complement of $\text{Halt}_0$.

(c) Prove that Total is neither c.e. nor co-c.e.

4. (The Busy Beaver function for running time.) For this problem and the next problem, restrict attention to Turing Machines over the alphabets $\Sigma = \Gamma = \{1, \$\}$. Define the running-time version of the Busy Beaver function $\text{BB}_{\text{steps}} : \mathbb{N} \to \mathbb{N}$ by

$$\text{BB}_{\text{steps}}(n) = \max\{m \geq 0 : \text{there is a Turing Machine } M \text{ with } n \text{ states such that } M(\epsilon) \text{ halts after exactly } m \text{ steps. }\}$$

(a) Prove that $\text{BB}_{\text{steps}}(n)$ is not computable by proving that otherwise $\text{Halt}_0^{TM}$ would be decidable. $\text{Halt}_0^{TM}$ is the version of $\text{Halt}_0$ that applies to algorithms which are specified as Turing machines.

(b) Suppose $f : \mathbb{N} \to \mathbb{N}$ is computable. Prove that $\text{BB}_{\text{steps}}(n)$ eventually dominates $f$, namely there is an $N$ such that $f(n) < \text{BB}_{\text{steps}}(n)$ for all $n > N$.

5. (Busy Beaver function for output value.) Use the same conventions as the previous problem. The Busy Beaver function $\text{BB} : \mathbb{N} \to \mathbb{N}$ is defined by

$$\text{BB}(n) = \max\{m \geq 0 : \text{There is a Turing Machine } M \text{ with } n \text{ states such that } M(\epsilon) \text{ halts with } m \text{ many 1's written on the tape.}\}$$
Suppose $f : \mathbb{N} \to \mathbb{N}$ is computable. Prove that $BB(n)$ eventually dominates $f$. Conclude that BB is not computable.