Math 160B - Winter 2022 - Homework 3
Due Wednesday, January 26, 11:00pm
(Hand in by uploading to Gradescope)

You can also find these problems in the PDF text.

1. Let R be a binary relation on Σ^*. Define S to be the set

$$S := \{w \in \Sigma^* : \exists v \in \Sigma^* (\langle v, w\rangle \in R)\}.$$

We call S the *projection* of R onto its second coordinate. Suppose that R is c.e. Prove that S is c.e.

2. Suppose R and S are k-relations on Σ^*. Define the *symmetric difference* $R \triangle S$ of R and S to equal $(R \setminus S) \cup (S \setminus R)$. Suppose that $R \triangle S$ is finite. Prove that R is decidable if and only if S is decidable. Prove that R is c.e. if and only if S is c.e.

3. Suppose that $R \subseteq \mathbb{N}$ is enumerated by an algorithm M in increasing order. In other words, M successively outputs $n_1, n_2, \ldots$ so that $R = \{n_1, n_2, n_3, \ldots\}$ and $n_i < n_{i+1}$ for each i. Prove that R is decidable. What if the assumption is weakened to the condition that $n_i \leq n_{i+1}$. Must R still be decidable?

4. Prove that a k-ary relation is computably enumerable if and only if it is the domain of a partial computable function.

5.Prove that a set is computably enumerable if and only if it is empty or is the range of a recursive function.

6. Prove Craig's Theorem:

(a) Suppose Γ is c.e. set of propositional formulas. Prove there is a decidable set Π of propositional formulas such that $\Gamma \models\!\!\!\!\!\dashv \Pi$.
(b) Suppose Γ is c.e. set of first-order sentences. Prove there is a decidable set Π of first-order sentences such that $\Gamma \models\!\!\!\!\!\dashv \Pi$.

Hint: The proofs for (a) and (b) are identical. One method of proof is to enumerate Γ as $A_1, A_2, \ldots$ and consider the formulas $A_1 \wedge A_2 \wedge \cdots \wedge A_k$.