Math 160B - Winter 2022 - Homework 2
Due Wednesday, January 19, 11:00pm
(Hand in by uploading to Gradescope)

1. Let T be the theory of linear orders from Example IV.89 using the language $L = \{<\}$. A model $\mathfrak{A}$ of T is called *well-founded* if there is no sequence $a_0, a_1, a_2, \ldots$ in $|\mathfrak{A}|$ such that $a_{i+1} <^{\mathfrak{A}} a_i$ holds for all $i \in \mathbb{N}$. Show that there is no set Γ of sentences over T that expresses the property of being well-founded. That is, there is no set Γ of sentences such that for all models $\mathfrak{A}$ of T, we have $\mathfrak{A} \models \Gamma$ if and only if $\mathfrak{A}$ is well-founded.

2. Let L be the language with a unary predicate symbol P (and, as always, equality). Let Γ be a set of sentences expressing that there infinitely many objects x satisfying $P(x)$ and infinitely many objects x satisfying $\neg P(x)$.

(a) Describe explicitly the formulas in Γ.
(b) Show that Γ is $\aleph_0$-categorial. Conclude that the theory axiomatized by Γ is complete.
(c) Show that Γ is not κ-categorical for $\kappa > \aleph_0$.

3. Give an example of a theory which is κ-categorical for all infinite κ.

4. A theory is called *categorical* if all models of T are isomorphic.

(a) Describe what must happen if T is categorical.

(b) Suppose T is a categorical theory over a finite language L. Prove that T is *finitely axiomatizable*, that is, that there is a finite set of sentences Γ such that $\Gamma \models\dashv T$.

5.Let $\Sigma = \{a\}$. Let $f : \Sigma^* \to \Sigma^*$ be the unary function so that for all $w \in \Sigma^*$,

$$f(w) = \begin{cases} \epsilon & \text{if the Riemann hypothesis is true} \\ a & \text{if the Riemann hypothesis is false.} \end{cases}$$

Prove that f is computable.

6.Let $\Sigma = \{1\}$. Let f be the unary function defined by

$$f(1^n) = \begin{cases} \epsilon & \text{if there is a run of } n \text{ consecutive 7's in the decimal expanson of } \pi \\ 1 & \text{otherwise.} \end{cases}$$

Prove that f is computable. (It is an open question whether there are arbitrarily long (finite) runs of 7's in the decimal expansion of π.)