1. Let $T$ be the theory of linear orders from Example IV.89 using the language $L = \{<\}$. A model $\mathfrak{A}$ of $T$ is called well-founded if there is no sequence $a_0, a_1, a_2, \ldots$ in $|\mathfrak{A}|$ such that $a_{i+1} <^\mathfrak{A} a_i$ holds for all $i \in \mathbb{N}$. Show that there is no set $\Gamma$ of sentences over $T$ that expresses the property of being well-founded. That is, there is no set $\Gamma$ of sentences such that for all models $\mathfrak{A}$ of $T$, we have $\mathfrak{A} \models \Gamma$ if and only if $\mathfrak{A}$ is well-founded.

2. Let $L$ be the language with a unary predicate symbol $P$ (and, as always, equality). Let $\Gamma$ be a set of sentences expressing that there infinitely many objects $x$ satisfying $P(x)$ and infinitely many objects $x$ satisfying $\neg P(x)$.

   (a) Describe explicitly the formulas in $\Gamma$.
   (b) Show that $\Gamma$ is $\aleph_0$-categorical. Conclude that the theory axiomatized by $\Gamma$ is complete.
   (c) Show that $\Gamma$ is not $\kappa$-categorical for $\kappa > \aleph_0$.

3. Give an example of a theory which is $\kappa$-categorical for all infinite $\kappa$.

4. A theory is called categorical if all models of $T$ are isomorphic.

   (a) Describe what must happen if $T$ is categorical.
   (b) Suppose $T$ is a categorical theory over a finite language $L$. Prove that $T$ is finitely axiomatizable, that is, that there is a finite set of sentences $\Gamma$ such that $\Gamma \models T$.

5. Let $\Sigma = \{a\}$. Let $f : \Sigma^* \to \Sigma^*$ be the unary function so that for all $w \in \Sigma^*$,

   $$f(w) = \begin{cases} 
   \epsilon & \text{if the Riemann hypothesis is true} \\
   a & \text{if the Riemann hypothesis is false} 
   \end{cases}$$

   Prove that $f$ is computable.

6. Let $\Sigma = \{1\}$. Let $f$ be the unary function defined by

   $$f(1^n) = \begin{cases} 
   \epsilon & \text{if there is a run of } n \text{ consecutive } 7\text{'s in the decimal expansion of } \pi \\
   1 & \text{otherwise.} 
   \end{cases}$$

   Prove that $f$ is computable. (It is an open question whether there are arbitrarily long (finite) runs of 7’s in the decimal expansion of $\pi$.)