Math 160A - Fall 2021

Midterm 2 — November 18, 2021 — Duration: ≈ 80 minutes (until 12:20pm)

Name:

PID:

Instructions:

- Check that you have all pages in the midterm (Problems 1-9).
- Write your name and student ID number above.
- Write your initials on the upper right corner of each page. This is in case exams are separated during scanning.
- Get a couple of pages of scratch pages from the instructor; they are handed in loose at the end of the exam. Work on scratch pages will **not** be checked, and does not count towards your grade. **Be sure to transfer enough work from scratch sheets to the answer sheets, so we can see how you arrived at your answers. Otherwise you may not receive credit for your answers.**
- Write your answers on the front sides of the pages; do not write on the back sides. Let us know if you need extra space.
- The problems are not in order of difficulty; you may wish to skip harder problems and come back to them later.

Academic Integrity Guidelines: **You must work this exam on your own. You may not use notes, textbooks, online resources, or resources of any kind.**

Initials:

1. Use unary predicates $Dog(x)$ and $Cat(x)$, the binary predicates $Likes(x, y)$ and $Knows(x, y)$, the constant symbol *John*, the unary function $Mother(x)$ and the equality sign = to express the following English sentences in first-order logic. $Dog(x)$ means "x is a dog" and $Cat(x)$ means "x is a cat". $Likes(x, y)$ means "x likes y" and $Knows(x, y)$ means "x knows y". $Mother(x)$ denotes the mother of x. Variables ranges over the universe containing all dogs, cats, and people.

(a) *John* knows a dog that does not like its mother.

(b) Every cat that *John* knows likes its mother.

(c) All cats have the same mother.

(d) No two dogs have the same mother.

(e) There is a cat that is not a mother.

(f) No cat is a mother.

(g) Every cat knows a dog that likes all cats.

Initials:

2. Work in the first-order language with a binary predicate symbol E and equality ($=$). Let A, B and C be the sentences

$$\begin{aligned} A &:= \forall x\, \forall y\, \forall z\, [E(x,y) \to E(y,z) \to E(x,z)]. \\ B &:= \forall x\, \exists y\, (x \neq y \wedge E(x,y)). \\ C &:= \exists y\, \forall x\, (x \neq y \to E(x,y)). \end{aligned}$$

Let $\Gamma = \{A, B\}$. Give, explicitly, a structure $\mathfrak{A}$ showing that $\Gamma \not\models C$. Describe $\mathfrak{A}$ by using set notation and tuple notation as appropriate.
Also, draw a picture of your structure.

Initials:

3. Give an example of formulas B and C such that

$$\exists x B \wedge C \not\vDash \exists x\,(B \wedge C). \tag{1}$$

Justify your answer by giving a structure $\mathfrak{A}$ and an object assignment σ showing that (1) holds. Specify $\mathfrak{A}$ explicitly. (Also see problem 4 at the bottom of this page.)

4. What extra assumption can you make about C that will make $\exists x B \wedge C \vDash \exists x\,(B \wedge C)$ be a valid logical implication?

Initials:

5. Suppose x is not free in B and prove that $\exists x\, B \vDash B$. Your proof should based on the definition of truth.

6. Let A be the formula

$$\exists x\,[x \leq y \wedge \forall y\,(P(y,z) \rightarrow \exists z(y \leq z))].$$

Parts (b)-(d) ask you to carry out substitutions: for your answers, give the formulas that result from the substitutions. u, v, w, x, y, z are distinct variables.

(a) In the formula A above, label all the free occurrences of variables.

(b) What is $A(g(0,u)/z)$?

(c) What is $A(f(v), g(0,u)/y, z)$?

(d) What is $A(g(w,0), f(v), g(0,u)/x, y, z)$?

(e) Give an alphabetic variant B of A so that $h(x,y,z)$ is subsitutable for y in B. For full credit, rename as few bound variables as possible.

7. For each of the following statements, say whether it is true for all formulas A or not. If it is not true for all formulas A, give an example of a formula A that makes the statement false.

(a) The term 0 is substitutable for x_1 in A. (0 is a constant symbol.)

(b) The term x_1 is substitutable for x_1 in A.

(c) The term x_2 is substitutable for x_1 in A.

Initials:

8. Use the language with a unary predicate symbol P and equality (=). Express the following statements in first-order logic.

(a) There is exactly one object x such that $P(x)$ holds.

(b) There are exactly two objects x such that $P(x)$ holds.

Initials:

9. Work in the language with symbols $0, 1, +, \cdot, \leq, =$ and write formulas that will be interpreted in the structure $\mathcal{N}$ consisting of the nonnegative integers with the nonlogical symbols having their usual interpretations:

$$\mathcal{N} = (\mathbb{N}, 0, 1, +, \cdot, \leq).$$

(a) Give a first-order formula $Mult(x, y)$ which expresses that y is a multiple of x.

(b) Give a first-order formula $CM(x, y, z)$ which expresses that z is a multiple of both x and y (that is, z is a common multiple of x and y). You may use "*Mult*" in your answer to avoid having to write out that formula again.

(c) Give a first-order formula $LCM(x, y, z)$ which expresses that z is a least common multiple of x and y. You may use "*Mult*" and "*CM*" in your answer.

(d) Give a first-order sentence which expresses that every pair x and y have a least common multiple. You may use "*Mult*" and "*CM*" and "*LCM*" in your answer.