Math 160A - Fall 2021
Final Exam — December 8, 2021 — Duration: 3 hours

Name:

PID:

Instructions:

- Check that you have all pages in the exam (Problems 1-12).
- Write your name and student ID number above.
- Write your initials on the upper right corner of each page. This is in case exams are separated during scanning.
- Get a couple of pages of scratch pages from the instructor; they are to be handed in loose at the end of the exam. Work on scratch pages will **not** be checked, and does not count towards your grade. **Be sure to transfer enough work from scratch sheets to the answer sheets, so we can see how you arrived at your answers. Otherwise you may not receive credit for your answers.**
- Write your answers on the front sides of the pages; do not write on the back sides. Let us know if you need extra pages.
- The problems might not be in order of difficulty; you may wish to skip harder problems and come back to them later.
- **There are two supplemental sheets with info about the PL and FO proof systems. These pages may also be used for scratch work, and are to be handed in. However, no work on these will count towards your grade.**

Academic Integrity Guidelines: **You must work this exam on your own. You may not use notes, textbooks, online resources, or resources of any kind.**

Initials:

1. Consider the following three propositional formulas:

(a) $p \to (q \to p) \to p$.

(b) $(p \to (q \to p)) \to p$.

(c) $((p \to q) \to p) \to p$.

For each one (please label your answers clearly):

(i) State whether it is "unsatisfiable", or "satisfiable but not a tautology" or "a tautology".

(ii) If it is not a tautology, give a truth assigment which falsifies it.

(iii) If it is a tautology, prove that it has a PL-proof. (You do not have to give an explicit PL proof; but you should not use the Completeness theorem.)

2. Use the unary predicate $Tall(x)$, the binary predicates $Likes(x, y)$ and $Knows(x, y)$ and the equality sign = to express the following English sentences in first-order logic. $Tall(x)$ means "x is tall", $Likes(x, y)$ means "x likes y" and $Knows(x, y)$ means "x knows y". Variables ranges over the universe of people.

(a) Everyone know someone who does not like them.

(b) No one likes everyone.

(c) Anyone who likes themselves is also liked by someone else.

(d) Everyone knows a tall person who likes no one.

(e) There is a tall person who knows exactly two people.

(f) There is someone who likes all tall people, but there is no one who likes everyone.

Initials:

3.a. Express $p \leftrightarrow (q \to r \wedge p)$ in CNF form and DNF form.

3.b. Let f be the Boolean function defined by $f(x_1, x_2, x_3, x_4) = \mathrm{T}$ if and only if at least three of x_1, x_2, x_3, x_4 are equal to T. Give a propositional formula that represents f (i.e., that defines f).

Initials:

4. Give a prenex formula which is logically equivalent to

$$\forall x \, \exists y \, P(x,y) \to \exists x \, \forall y \, P(x,y).$$

5. Indicate, for (a)-(d), whether they are true or false statements. To be true, it must always be true. A is a formula, t is a term, c is a constant symbol, f is a function symbol.

(a) If t is a closed term, then t is substitutable for x in A.

(b) The term $f(x,x)$ is substitutable for x in A.

(c) If t is substitutable for x in A, then $A(t/x)$ is a formula.

(d) If $A(t/x)$ is a formula, then t is substitutable for x in A.

Initials:

6. Let $Maj^3(x, y, z)$ be the 3-ary majority function.

(a) Prove that $\{Maj^3, \oplus\}$ is not an adequate set of propositional connectives.

(b) Prove that $\{\oplus, \rightarrow\}$ is an adequate set of propositional connectives.

Initials:

7. Show that $\{\forall x\, \exists y\, P(x, y),\, \neg P(c, z)\}$ is inconsistent by giving an explicit FO proof (or explicit FO proofs).

Initials:

8.a. Prove there exists an FO-proof of $\forall x\,\forall y\,(P(x) \to Q(y)) \to \exists x\,P(x) \to \forall y\,Q(y)$.

8.b. Prove there exists an FO-proof of $\exists y\,\forall x\,Q(x,y) \vdash \forall x\,\exists y\,Q(x,y)$. [Hint: It is optional, but you may wish to use the Theorem on Constants).

9. The "Proof-by-Cases" principle for FO (see the supplemental pages) assumes that A is a sentence. Give an example of how the "Proof-by-Cases" principle can fail when A is a formula by giving an example of a formula A and a sentence B such that $A \vdash B$ and $\neg A \vdash B$ but $\nvdash B$. Explain why your example works.

Initials:

10. Let the language L contain the constant symbol c, unary function symbol f. Give an example of an L-structure $\mathfrak{A}$ which satisfies

$$\Gamma = \{\forall x\, \forall y\, (x \neq y \to f(x) \neq f(y)),\ \forall x\, f(x) \neq c\}.$$

Describe the universe of $\mathfrak{A}$ and the interpretations of the two non-logical symbols explicitly (preferably in set notation).

Initials:

11. Work in the language L with the symbol $=$ and no non-logical symbols. We say that "Γ has a model of cardinality n" if there is a structure $\mathfrak{A}$ so that $\mathfrak{A} \models \Gamma$ and its universe $|\mathfrak{A}|$ has size n.

(a) For each $n \geq 1$, describe a satisfiable L-sentence A_n so that every model of A has cardinality n. (It is fine to use "big-and" and "big-or" notations.)

(b) Give an example of a set Γ of L-sentences which has models of cardinality $2n$ for all integers $n > 0$, but does not have any model of cardinality $2n + 1$ for any integer $n \geq 0$. Call this the "even cardinality finite models" property.

12. This revisits problem 11. Give a language L^{+} and a finite set Π of L^{+}-sentences so that the "even cardinality finite models" property of part (b) of problem 11 holds for Π. Explain why your example works.

Supplemental notes for Math 160A Final Exam (Winter 2021).
These are to be handed in, but do not count towards your grade.

Constructing PL-proofs

Logical axioms: PL1: $A \to (B \to A)$
PL2: $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$
PL3: $\neg A \to (A \to B)$
PL4: $(\neg A \to A) \to A$

Modus Ponens: $\dfrac{A \to B \qquad A}{B}$

Abbreviations: $A \vee B$, $A \wedge B$ and $A \leftrightarrow B$ abbreviate, respectively,
$\neg A \to B$, $\neg(A \to \neg B)$ and $(A \to B) \wedge (B \to A)$.

Deduction Theorem: $\Gamma \vdash A \to B$ iff $\Gamma, A \vdash B$.

Proof by Contradiction:

$\Gamma \vdash A$ iff $\Gamma \cup \{\neg A\}$ is inconsistent.

$\Gamma \vdash \neg A$ iff $\Gamma \cup \{A\}$ is inconsistent.

Proof by Cases: If $\Gamma, A \vdash B$ and $\Gamma, \neg A \vdash B$, then $\Gamma \vdash B$.

Derived (admissible) rules of inference:

Modus Tollens: $\dfrac{A \to B \qquad \neg B}{\neg A}$

Hypothetical Syllogism: $\dfrac{A \to B \qquad B \to C}{A \to C}$

Axioms and Inference Rules for FO-proofs

PL1: $A \to B \to A$

PL2: $(A \to B \to C) \to (A \to B) \to (A \to C)$

PL3: $\neg A \to A \to B$

PL4: $(\neg A \to A) \to A$

EQ1: $x = x$

EQ2: $x = y \to y = x$

EQ3: $x = y \to y = z \to x = z$

EQ$_f$: $y_1 = z_1 \to \cdots \to y_k = z_k \to f(y_1, \ldots, y_k) = f(z_1, \ldots, z_k)$

EQ$_P$: $y_1 = z_1 \to \cdots \to y_k = z_k \to P(y_1, \ldots, y_k) \to P(z_1, \ldots, z_k)$

Universal Instantiation (UI): $\forall x\, A(x) \to A(t)$.

Modus Ponens: $\dfrac{A \qquad A \to B}{B}$.

Generalization (Gen): $\dfrac{C \to A}{C \to \forall x\, A}$ $\quad$ (x not free in C)

$A \vee B$, $A \wedge B$, and $\exists x\, A$ abbreviate $\neg A \to B$, $\neg(A \to \neg B)$ and $\neg \forall x \neg A$.

Constructing FO-proofs

Derived rules:

Propositional rules: Modus Tollens, Hypothetical Syllogism and more generally Tautological Implication (TAUT).

Generalization: $\dfrac{A}{\forall x\, A}$

If t is substitutable for x in A:

UI Rule: $\dfrac{\forall x\, A(x)}{A(t)}$ $\qquad$ **Substitution:** $\dfrac{A(x)}{A(t)}$

For Deduction/Contradiction/By Cases, assume A is a sentence.

Deduction Theorem: $\Gamma \vdash A \to B$ iff $\Gamma, A \vdash B$.

Proof by Contradiction:

$\Gamma \vdash A$ iff $\Gamma \cup \{\neg A\}$ is inconsistent.

$\Gamma \vdash \neg A$ iff $\Gamma \cup \{A\}$ is inconsistent.

Proof by Cases: If $\Gamma, A \vdash B$ and $\Gamma, \neg A \vdash B$, then $\Gamma \vdash B$.

Theorem on Constants. c is a new constant symbol, not in Γ, A, B.

For $A(x)$ a formula, $\Gamma \vdash \forall x\, A(x)$ iff $\Gamma \vdash A(c)$.

If $\exists x\, A(x)$ is a sentence, $\Gamma \cup \{\exists x\, A(x)\} \vdash B$ iff $\Gamma \cup \{A(c)\} \vdash B$